Basis Theorems: Reverse mathematics and Weihrauch reductions

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Basis theorems from reverse mathematics

RCA_0 is a weak subsystem of second order arithmetic that includes axioms for recursive comprehension and induction restricted to Σ^0_1 formulas.

Basis theorems from reverse mathematics

RCA_0 is a weak subsystem of second order arithmetic that includes axioms for recursive comprehension and induction restricted to Σ^0_1 formulas.

Theorem: (RCA_0) The following are equivalent:

(1) $\mathsf{I}\Sigma^0_2$, induction for Σ^0_2 formulas.

(2) Color basis: If $f : \mathbb{N} \to k$ then there is a set consisting of those $j < k$ appearing infinitely often in the range of f. [1]

(3) Vector space basis: If V is a countable vector space of bounded dimension, then V has a basis. [3]

(4) e-matroid basis: If M is an e-matroid of bounded dimension, then M has a basis. [3]

In their article
On the Weihrauch degree of the additive Ramsey theorem
Computability (2024) [4],
Arno Pauli, Cécilia Pradic, and Giovanni Soldà noted that:

many principles equivalent to $\mathsf{I}\Sigma^0_2$ over RCA_0 have associated Weihrauch principles related to $\mathsf{TC}^*_{\mathbb{N}}$, $(\mathsf{LPO}')^*$ or $\mathsf{TC}^*_{\mathbb{N}} \times (\mathsf{LPO}')^*$.

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$\mathsf{TC}_{\mathbb{N}}$: Totalization of closed choice on $\mathbb{N}$. Given $e : \mathbb{N} \to \mathbb{N}$ choose n such that either $n \notin \mathrm{Range}(e)$ or $\mathbb{N} = \mathrm{Range}(e)$.

LPO': Jump of the limited principle of omniscience. Given infinite binary strings $\langle p_0, p_1, \ldots \rangle$, apply LPO to $\lim_i p_i$.

$\times$ denotes parallel product
$(\ )^*$ denotes finite parallel products

The previously mentioned basis theorems exhibit this behavior. As Weihrauch principles:

- Matroid basis is equivalent to $\mathsf{TC}_{\mathbb{N}}^{*} \times (\mathsf{LPO}')^{*}$ [3]
- Vector space basis is equivalent to $\mathsf{TC}_{\mathbb{N}}^{*} \times (\mathsf{LPO}')^{*}$ [3]
- Color basis is equivalent to $(\mathsf{LPO}')^{*}$ [1] (with help)

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In general, given

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- a spanning set,

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we would like to find a basis.

How can we reformulate basis problems to modify the Weihrauch strength?

e-matroid

An e-matroid consists of a countable set M and an enumeration $e : \mathbb{N} \to [M]^{<\mathbb{N}}$ of the finite dependent sets of M.
Properties:

- $\emptyset$ is independent.
- Finite supersets of dependent sets are dependent.
- If X and Y are independent sets and $|X| < |Y|$, then for some $y \in Y$ the set $X \cup \{y\}$ is independent.

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If $\text{Range}(e)$ includes all the sets of size b, then the dimension (rank) of the matroid (M, e) is bounded by b.

A basis of a matroid is a maximal independent set.

Let EMB_+ denote the problem which given input (M, e) and b, outputs a basis for (M, e).

As noted above $\text{EMB}_+ \equiv_{\text{W}} \text{TC}^*_{\mathbb{N}} \times (\text{LPO}')^*$ [3]

Reformulating EMB_+

We should be able to revise EMB_+ to get principles of lower Weihrauch strength. EMB_+ inputs a bound on the dimension.

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Exact dimension reformulation

Let $\mathsf{EMB}_{\mathsf{rank}}$ denote the problem which given input (M, e) and the exact rank d of M, outputs a basis.

Using a bijection between M^d and $\mathbb{N}$ we can find an enumeration of the dependent subsets of M of size d. Any set not enumerated is a basis.

Conjecture: $\mathsf{EMB}_{\mathsf{rank}} \equiv_{\mathsf{W}} \mathsf{C}_{\mathbb{N}}$.

Conjecture: RCA_0 proves $\mathsf{EMB}_{\mathsf{rank}}$.

Perhaps $\mathsf{RCA}_0^* \vdash \mathsf{EMB}_{\mathsf{rank}} \leftrightarrow \mathsf{I}\Sigma^0_1$.

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Spanning set reformulation

Defn: A set S is a spanning set for a matroid M if for every $x \in M - S$ there is an independent set $I \subset S$ such that $I \cup \{x\}$ is dependent.

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Let EMB_{span} denote the problem which given input (M, e) and a finite spanning set S of M, outputs a basis.

For each subset of S, a single use of LPO can determine if the subset is in the range of e. If it's independent and maximal, then it is a basis.

Conjecture: $\text{EMB}_{\text{span}} \equiv_{\text{W}} \text{LPO}^*$.

Conjecture: RCA_0 proves EMB_{span}.

Perhaps $\text{RCA}_0^* \vdash \text{EMB}_{\text{span}} \leftrightarrow \text{I}\Sigma^0_1$.

Color basis recalled

Given $f : \mathbb{N} \to k$, find the color basis, that is, the set of $j < k$ such that j appears infinitely often in Range(f).

Theorem: (RCA_0) $\mathsf{I}\Sigma^0_2$ if and only if every $f : \mathbb{N} \to k$ has a color basis. [1]

Let CB_+ denote the problem which given input $f : \mathbb{N} \to k$ outputs a color basis for f.

Theorem: $\mathsf{CB}_+ \equiv_{\mathrm{W}} (\mathsf{LPO}')^*$ [1]

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Can we reformulate color basis to increase the Weihrauch strength to $\mathsf{TC}^*_{\mathbb{N}} \times (\mathsf{LPO}')^*$?

The color basis is a subset of $[0, k)$, so k is acting like a spanning set.

Color basis reformulated

Defn: A function $f : \mathbb{N} \to \mathbb{N}$ has at most b color basis elements if, with at most b exceptions, every element of Range(f) appears only finitely many times.

Conjecture: (RCA_0) $\mathsf{I}\Sigma^0_2$ if and only if every $f : \mathbb{N} \to \mathbb{N}$ that has at most b color basis elements has a color basis.

Let $\mathsf{CB}_{\mathsf{size}}$ denote the problem which given b and a function $f : \mathbb{N} \to \mathbb{N}$ that has at most b color basis elements, outputs a color basis for f.

Conjecture: $\mathsf{CB}_{\mathsf{size}} \equiv_{\mathsf{W}} \mathsf{TC}^*_{\mathbb{N}} \times (\mathsf{LPO}')^*$.

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What about the similar problem with exact size information? Is it related to $\mathsf{TC}^*_{\mathbb{N}}$? Is it equivalent to $\mathsf{I}\Sigma^0_2$?

What about vector space results?

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