Hindman's Theorem, Ultrafilters, and Reverse Mathematics

ASL in Baltimore

January 18, 2003

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Overview:

- Hindman's theorem
- Reverse mathematics / computability
- Ultrafilter version of Hindman's theorem
- A computable restriction of Hindman's theorem
- Stone representation theorem
- Glazer's proof of Hindman's theorem

$\mathsf{FS}(X) :=$ all sums of finite subsets of X

Example: Suppose $X = \{1, 2, 5\}$.

Then $\mathsf{FS}(X) = \{1, 2, 3, 5, 6, 7, 8\}$

No repeating!

Theorem 1 (Hindman's Theorem [?]). *Given $G \subseteq \mathbb{N}$, there is an infinite set $X \subseteq \mathbb{N}$ such that $\mathsf{FS}(X) \subseteq G$ or $\mathsf{FS}(X) \subseteq G^c$.*

Example: Suppose G is the set of natural numbers that have an even number of factors of 2 in their prime factorization.

0	1	2	3	4	5	6	7	8	16
G	G		G	G	G		G		G

Candidates for X:

1. G doesn't work, because $3 + 5 = 8$.

2. G^c doesn't work, because $2 + 6 + 8 = 16$.

3. $X = \{2^1, 2^3, 2^5, 2^7, \dots\}$ works! Every nonrepeating finite sum is in G^c.

Sometimes it's hard to find X.

Reverse mathematics can measure the proof-theoretic and computability-theoretic strength of theorems.

Subsystems of second order arithmetic:

RCA_0: (recursive comprehension)

- ordered semi-ring axioms
- induction for Σ^0_1 formulas
- existence axioms for relatively computable sets
- model: ω and the computable sets

ACA_0: (arithmetical comprehension)

- RCA_0 plus existence axioms for sets defined by formulas containing number quantifiers (no set quantifiers)
- model: ω and the arithmetically definable sets

Examples of Reverse Mathematics

Theorem 2. (D. Brown [**?**]) RCA_0 *proves that the following are equivalent:*

1. ACA_0.
2. *Suppose F is a closed subset of Cantor space. Then there is a countable subset C of F such that each element of F is a limit of elements of C.*

Theorem 3. (Blass, Hirst, Simpson [**?**]) ACA_0^+ *proves Hindman's theorem.*

Theorem 4. (Blass, Hirst, Simpson [**?**]) *Hindman's theorem proves* ACA_0.

Ultrafilters

An ultrafilter on the power set of $\mathbb{N}$ satisfies:

1. $\emptyset \notin U$
2. if $X_1, X_2 \in U$ then $X_1 \cap X_2 \in U$ (closed under intersections)
3. $\forall X \in U\ \forall Y \in F\ (X \subseteq Y \to Y \in U)$ (closed under supersets)
4. $\forall X \in F\ (X \in U \vee X^c \in U)$

Example: $[2] = \{X \subseteq \mathbb{N} \mid 2 \in X\}$ is called the principal ultrafilter generated by 2.

Nonexample: Cof$= \{X \subseteq \mathbb{N} \mid X^c \text{ is finite}\}$ is a filter, but not an ultrafilter. We could use Zorn's lemma to extend it to a nonprincipal ultrafilter.

Downward Translations

For $X \subseteq \mathbb{N}$ and $m \in \mathbb{N}$, let

$$X - m = \{y \in \mathbb{N} \mid y + m \in X\}$$

n	0	1	2	3	4	5	6	7	8
$n \in X$	0	1	1	0	0	0	1	1	1
$n \in X - 2$	1	0	0	0	1	1	1	?	?

An ultrafilter U is almost downward translation invariant if

$$\forall X \in U \ \exists x \in X \ (x \neq 0 \wedge X - x \in U)$$

Theorem 5. (Hindman [**?**]) *Assuming CH, Hindman's theorem holds if and only if there is an almost downward translation invariant ultrafilter on the subsets of* $\mathbb{N}$.

Reformulation in countable setting

A countable field of sets (Boolean algebra of sets) is a collection of subsets of $\mathbb{N}$ which is closed under intersection, (finite) union, and complement.

Note: Any ultrafilter on a countable field of sets is a countable set.

A downward translation algebra is a field of sets that is closed under downward translations.

Examples:

1. $\{\mathbb{N}, \emptyset\}$ is a downward translation algebra.

2. The computable sets form a countable downward translation algebra.

For any set G, let $\langle G \rangle$ denote the downward translation algebra generated by G.

Example: $\langle \text{evens} \rangle = \{\text{evens}, \text{odds}, \mathbb{N}, \emptyset\}$

Typical elements of $\langle G \rangle$ can be written in the form:

$$(G - m_{1,1} \cap G - m_{1,2} \cap \cdots \cap G^c - m_{1,j_1}) \cup \dots$$

$$\cdots \cup (G - m_{k,1} \cap G - m_{k,2} \cap \cdots \cap G^c - m_{k,j_k})$$

RCA_0 can prove that $\langle G \rangle$ exists.

P-theorem 6. (RCA_0) *Fix* $G \subseteq N$*. If there is an almost downward translation invariant ultrafilter on the downward translation algebra* $\langle G \rangle$*, then Hindman's Theorem holds for* G*.*

Sketch: Suppose U is an a.d.t.i.u.f. on $\langle G \rangle$.
Consider the case when $G \in U$.

Let $X_0 = G$.
Pick $x_0 \in X_0$ such that $X_0 - x_0 \in U$.

Let $X_1 = X_0 \cap X_0 - x_0$. Note $X_1 \in U$.
Pick $x_1 \in X_1$ such that
$x_0 < x_1$ and $X_1 - x_1 \in U$.

Let $X_2 = X_1 \cap X_1 - x_1$.
Pick $x_2 \in X_2$ such that ...

$G = X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$, so
$\{x_0, x_1, x_2, \ldots\} \subseteq G$.

$x_1 \in X_1 \subseteq X_0 - x_0$, so $x_0 + x_1 \in X_0 = G$.

Similarly, $x_1 + x_2 \in X_1 \subseteq G$.

Also, $x_1 + x_2 \in X_1 \subseteq X_0 - x_0$, so
$x_0 + x_1 + x_2 \in X_0 = G$.

$x_2 \in X_1 \subseteq X_0 - x_0$, so $x_0 + x_2 \in X_0 = G$.

P-theorem 7. (RCA_0) *If $\langle G \rangle$ contains no singletons, then Hindman's Theorem holds for G.*

Sketch:
Suppose $\langle G \rangle$ contains no singletons.

Principal ultrafilters on $\langle G \rangle$ are peculiar.

Consider $U = \{X \in \langle G \rangle \mid 0 \in X\}$.

Let $X \in U$. Pick an $x \in X$ such that $x \neq 0$.

Then $0 \in X - x$, so $X - x \in U$.

So U is an a.d.t.i.u.f. on $\langle G \rangle$.

By Theorem 6, Hindman's theorem holds for G.

An extension:

P-theorem 8. ($\mathsf{RCA}_0 + \Sigma^0_2$ induction) *If $\langle G \rangle$ doesn't contain all the singletons, then Hindman's Theorem holds for G.*

Consequences:

Corollary 9. *If G is computable and $\langle G \rangle$ doesn't contain all the singletons, then there is a computable set X satisfying Hindman's theorem for G.*

Corollary 10. *Let G be a computable set such that $0'$ is computable from every X satisfying Hindman's theorem for G. Then $\langle G \rangle$ contains all the singletons.*

Observation: Sometimes it's good to look at *all* the ultrafilters of a Boolean algebra.

Thm 11 (Stone Representation Thm). *Every Boolean algebra is isomorphic to a field of sets.*

Idea: Given a Boolean algebra (described algebraically) assign a set to each element.

How are ultrafilters used in assigning sets?

How can the proof be adapted to reverse mathematics?

Stone's Theorem for a particular finite boolean algebra

The algebra:

Elements: $\emptyset$, 1, a_1, a_2, a_3, a_4, a_5, a_6

Secret Key:

$\emptyset$	$\emptyset$
1	$\{1,2,3\}$
a_1	$\{1,2\}$
a_2	$\{1,3\}$
a_3	$\{2,3\}$
a_4	$\{1\}$
a_5	$\{2\}$
a_6	$\{3\}$

Operation fragments:

$\cap$	$\emptyset$	1	a_1	$a_2 \dots$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset \dots$
1	$\emptyset$	1	a_1	$a_2 \dots$
a_1	$\emptyset$	a_1	a_1	$a_4 \dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$\cup$	$\emptyset$	1	a_1	$a_2 \dots$
$\emptyset$	$\emptyset$	1	a_1	$a_2 \dots$
1	1	1	1	$1 \dots$
a_1	a_1	1	a_1	$1 \dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

a	$\emptyset$	1	a_1	$a_2 \dots$
a^c	1	$\emptyset$	a_6	$a_5 \dots$

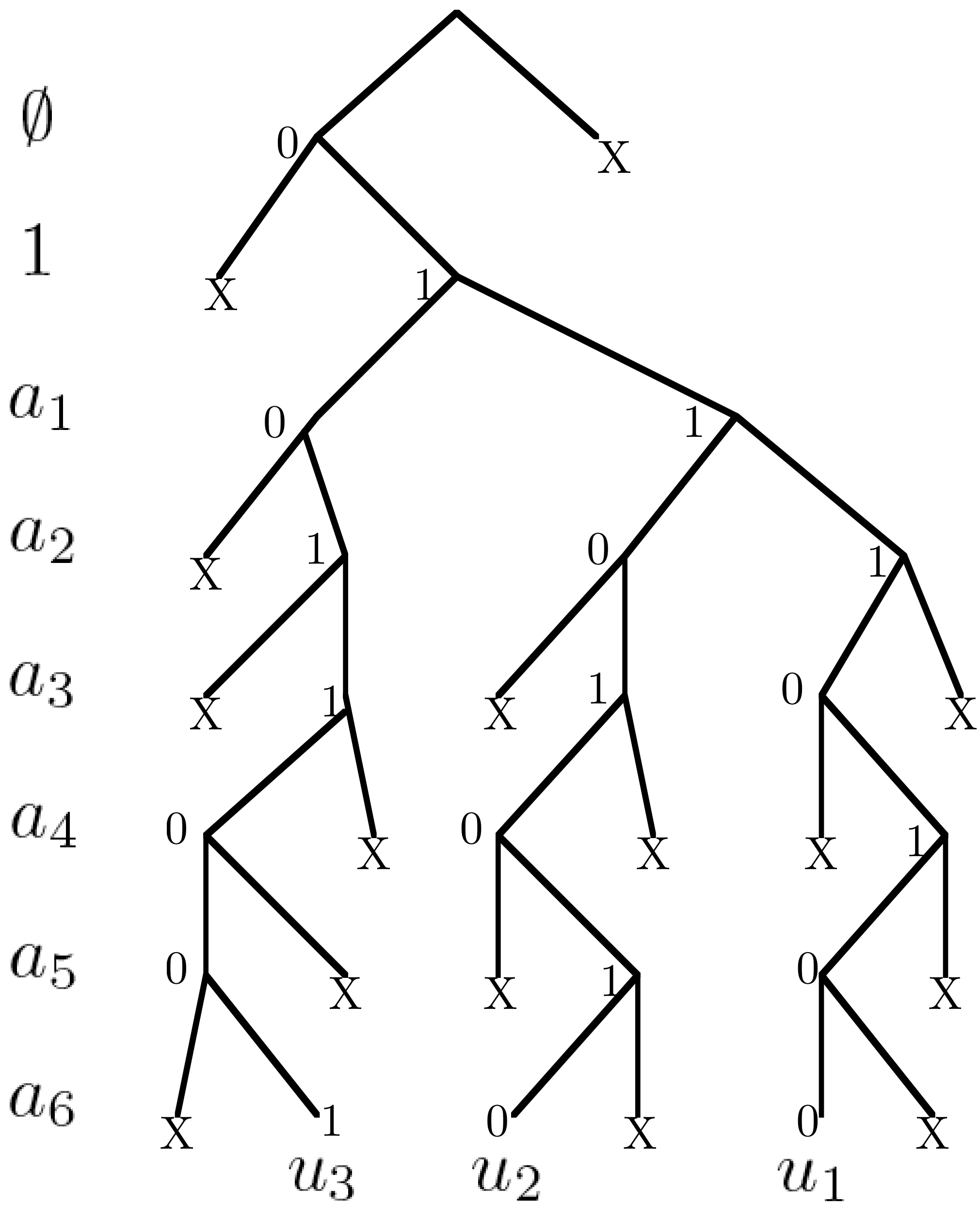
The ultrafilters on the algebra:

	u_1	u_2	u_3
$\emptyset$	0	0	0
1	1	1	1
a_1	1	1	0
a_2	1	0	1
a_3	0	1	1
a_4	1	0	0
a_5	0	1	0
a_6	0	0	1

The isomorphism:

$\emptyset$	$\emptyset$
1	$\{u_1, u_2, u_3\}$
a_1	$\{u_1, u_2\}$
a_2	$\{u_1, u_3\}$
a_3	$\{u_2, u_3\}$
a_4	$\{u_1\}$
a_5	$\{u_2\}$
a_6	$\{u_3\}$

If the algebra is infinite, then there may be uncountably many ultrafilters. It's more convenient to think of a tree.



The set of ultrafilters assigned to 1 (for example) may be may be uncountable.

P-theorem 12. (ACA_0) *Every countable Boolean algebra is isomorphic to a field of sets.*

Ideas from proof:

- The ultrafilters on the algebra form a closed subset of Cantor space; call it F.
- By Brown's theorem, ACA_0 proves the existence of a countable subset $C \subseteq F$ such that each element of F is a limit of elements of C.
- The sets $X_a = \{u \in C \mid a \in u\}$ form a field of sets isomorphic to the original algebra.

Corollary 13. *Every arithmetical countable boolean algebra is arithmetically isomorphic to an arithmetical field of sets.*

Glazer's proof of Hindman's theorem uses the space of ultrafilters on the power set of $\mathbb{N}$. (See page 449 of [?] or page 148 of [?].)

Outline of Glazer's proof:

- Define addition on ultrafilters.
- Prove that addition is associative and right continuous.
- Use compactness of the ultrafilter space to prove the existence of an idempotent ultrafilter ($u + u = u$).
- Show that idempotent ultrafilters are almost downward translation invariant.

Glazer's addition:

$X \in u + v$ if and only if

$$\exists Y \in v \ \forall y \in Y (X - y \in u)$$

Example:

$$X \in [2] + [3]$$

Can Glazer's proof be executed in ACA_0?

Pitfalls and responses:

Ultrafilters on the power set of $\mathbb{N}$ are uncountable.

> Use ultrafilters on a countable downward translation algebra.

The space of ultrafilters is uncountable.

> It's a countably encodable subset of Cantor space.

The proof of the existence of idempotents uses Zorn's lemma.

> Zorn's lemma can be avoided in a complete separable metric space.

The proof of right continuity the fact that every subset of $\mathbb{N}$ is in the ultrafilter.

> Uh-oh.

Adaptability

The general connection between ultrafilters and Ramsey theory:

Thm 14. (Hindman, page 148 of [**?**])

Let $\mathcal{G}$ be a family of nonempty subsets of X. The following are equivalent:

1. If X is finitely colored there exists a monochromatic $G \in \mathcal{G}$.
2. There exists an ultrafilter $\mathcal{A}$ on X such that, for all $A \in \mathcal{A}$, $A \supseteq G$ for some $G \in \mathcal{G}$.

The ultrafilter versions of Ramsey-style theorems can have interesting computability theoretic and reverse mathematical analogs.

References

[1] A. Blass, J. Hirst, and S. Simpson. Logical Analysis of Some Theorems of Combinatorics and Topological Dynamics. In: *Logic and Combinatorics* (Editor: S. Simpson), Contemporary Mathematics, **65**:125–156, 1987.

[2] D.K. Brown. Notions of closed subsets of a complete separable metric space in weak subsystems of second order arithmetic. In: *Logic and Computation* (Editor: W. Sieg), Contemporary Mathematics, **106**:39–50, 1990.

[3] W.W. Comfort. Ultrafilters: Some old and some new results. *Bull. Amer. Math. Soc.*, **83**:417–455, 1977.

[4] R. Graham, B. Rothschild, J. Spencer. *Ramsey theory*, Wiley-Interscience, New York, 1980.

[5] P.R. Halmos. *Lectures on Boolean algebras*, Van Nostrand, Princeton, 1963.

[6] N. Hindman. The existence of certain ultrafilters on N and a conjecture of Graham and Rothschild. *Proc. Amer. Math. Soc.*, **36**:341–346, 1972.

[7] N. Hindman. Finite sums from sequences within cells of a partition of N. *J. Combin. Theory Ser. A*, **17**:1–11, 1974.

[8] S.G. Simpson. *Subsystems of second order arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999.

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