Notes on Hindman's Theorem, Ultrafilters, and Reverse Mathematics

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Abstract

These notes give additional details for results presented at a talk in Baltimore on January 18, 2003.

1 Terminology

A *c*ountable field of sets is a countable collection of subsets of $\mathbb{N}$ which is closed under intersection, union, and relative complementation. We use X^c to denote $\{x \in \mathbb{N} \mid x \notin X\}$, the complement of X relative to $\mathbb{N}$.

Given a set $X \subseteq \mathbb{N}$ and an integer $n \in \mathbb{N}$, we use $X - n$ to denote the set $\{x - n \mid x \in X \wedge x \geq n\}$, the translation of X by n. A *downward translation algebra* is countable field of sets which is closed under translation.

Given a set $G \subseteq \mathbb{N}$, $\mathsf{RCA_0}$ suffices to prove the existence of the set $\langle G \rangle$ of all finite unions of finite intersections of translations of G and complements of translations of G. $\mathsf{RCA_0}$ can also prove that $\langle G \rangle$ is a downward translation algebra. Note that in $\mathsf{RCA_0}$, we encode $\langle G \rangle$ as a countable sequence of countable sets, and that each set in $\langle G \rangle$ may be repeated many times in the sequence.

An ultrafilter on a countable field of sets F is a subset $U \subseteq F$ satisfying the following four properties.

1. $\emptyset \notin U$.

2. if $X_1, X_2 \in U$ then $X_1 \cap X_2 \in U$.

3. $\forall X \in U\ \forall Y \in F\ (X \subseteq Y \to Y \in U)$.

4. $\forall X \in F\ (X \in U \vee X^c \in U)$.

An ultrafilter U is an almost downward translation invariant ultrafilter if $\forall X \in U\ \exists x \in X\ (x \neq 0 \wedge X - x \in U)$.

Comment: Note that the restriction to fields of sets other than the full power set of $\mathbb{N}$ affects the formulation of the last definition. An alternate definition of almost downward translation invariant ultrafilter is one in which $\{x \in \mathbb{N} \mid A - x \in U\} \in U$ whenever $A \in U$. Lemma 3.1 of [3] shows that the two definitions are equivalent for ultrafilters on the full power set of $\mathbb{N}$. Because countable fields of sets fail to contain many subsets of $\mathbb{N}$, the definitions are not interchangeable in the countable setting. The relative sparseness of the fields of sets considered here is a primary factor contributing to the technical difficulty of adapting ultrafilter based arguments to the reverse mathematical setting.

Lemma 1. (RCA_0) *If U is an almost downward translation invariant ultrafilter and X is an element of U, then the set $\{x \in X \mid X - x \in U\}$ is unbounded.*

Proof. Suppose by way of contradiction that x is the largest number in X such that $X - x \in U$. Since $X - x \in U$, there is a $y \in X - x$ such that $y \neq 0$ and $(X - x) - y \in U$. Since $y \in X - x$, we have $x + y \in X$. Additionally, $(X-x)-y = X-(x+y)$, so we have $x+y \in X$, $x+y > x$, and $X-(x+y) \in U$, contradicting our choice of x. □

If $X \subseteq \mathbb{N}$, then the notation $\mathsf{FS}(X)$ denotes the set of all nonrepeating sums of nonempty finite subsets of X. For example, $\mathsf{FS}(\{1, 2, 5\}) = \{1, 2, 3, 5, 6, 7, 8\}$.

Theorem 2 (Hindman's Theorem). *If $\langle C_i \rangle_{i<m}$ is a partition of $\mathbb{N}$ into m disjoint subsets, then there is an infinite set $X \subseteq \mathbb{N}$ and a value $c < m$ such that $\mathsf{FS}(X) \subseteq C_c$.*

Proof. See [4]. □

The set X in the statement of Theorem 2 is called an infinite homogeneous set for the partition. Given any set $G \subseteq \mathbb{N}$, we refer to the statement of Theorem 2 with $C_0 = G$ and $C_1 = G^c$ as Hindman's Theorem for G.

2 Ultrafilters and Hindman's Theorem

Theorem 3.3 of [3] states that assuming the continuum hypothesis, Hindman's Theorem holds if and only if there is an almost downward translation invariant ultrafilter on the field of all subsets of $\mathbb{N}$. The easy direction of this theorem can be reformulated and proved in the countable setting as follows.

Theorem 3. (RCA_0) *Fix $G \subseteq N$. If there is an almost downward translation invariant ultrafilter on the downward translation algebra $\langle G \rangle$, then Hindman's Theorem holds for G.*

Proof. Assume RCA_0, suppose that $G \subseteq N$, and let U be an almost downward translation invariant ultrafilter on $\langle G \rangle$. Define a decreasing sequence of sets $\langle X_i \rangle$ and an increasing sequence of integers $\langle x_i \rangle$ as follows. Let X_0 be whichever of G and G^c is in U. Let $x_0 = 0$. Suppose that $X_n \in U$ and x_n have been chosen. Since U is an almost downward translation invariant ultrafilter, by Lemma 1 we can find a least $x_{n+1} \in X_n$ such that $x_{n+1} > x_n$ and $X_n - x_{n+1} \in U$. Let $X_{n+1} = X_n \cap (X_n - x_{n+1})$.

We will show that $\mathsf{FS}(\langle x_n \rangle_{n>0}) \subseteq X_0$, thereby proving that Hindman's Theorem holds for G. Let $x_{i_1}, \ldots, x_{i_k}$ be a finite sequence of elements of $\langle x_n \rangle_{n>0}$. Note that $i_k > 0$, so X_{i_k-1} exists, and $x_{i_k} \in X_{i_k-1}$. Treating this as the base case in an induction, we have $\sum_{m=k}^{k} x_{i_m} \in X_{i_k-1}$. For the induction step, suppose that $\sum_{m=j+1}^{k} x_{i_m} \in X_{i_{j+1}-1}$. Since $i_j \leq i_{j+1} - 1$, $\sum_{m=j+1}^{k} x_{i_m} \in X_{i_j}$. Since $i_j \geq 1$, by the definition of X_n, $X_{i_j} = X_{i_j-1} \cap (X_{i_j-1} - x_{i_j})$, so $\sum_{m=j+1}^{k} x_{i_m} \in X_{i_j-1} - x_{i_j}$. That is, $\sum_{m=j}^{k} x_{i_m} \in X_{i_j-1}$, completing the induction step. By induction on quantifier-free formulas, we have shown that $\sum_{m=1}^{k} x_{i_m} \in X_{i_1-1} \subseteq X_0$. Generalizing, we can conclude that every finite sum of elements of $\langle x_n \rangle_{n>0}$ is an element of X_0, witnessing that Hindman's theorem holds for G. $\square$

3 A special case of Hindman's Theorem

The main result of this section is that RCA_0 proves Hindman's Theorem for those partitions G such that $\langle G \rangle$ does not contain all the singleton sets.

Lemma 4. (RCA_0) *If the downward translation algebra $\langle G \rangle$ contains no singletons, then Hindman's Theorem holds for G.*

Proof. Suppose $\langle G \rangle$ contains no singletons. Let U be the principal ultrafilter generated by 0, so $U = \{X \in \langle G \rangle \mid 0 \in X\}$. Pick $X \in U$. Since X is not a singleton, there is an $x \neq 0$ such that $x \in X$. Since $x \in X$, we have $0 \in X - x$, so $X - x \in U$. Thus U is an almost downward translation invariant ultrafilter. By Theorem 3, Hindman's theorem holds for G. $\square$

Alternate Proof. Suppose $\langle G \rangle$ contains no singletons. Since $\langle G \rangle = \langle G^c \rangle$, we may relabel as needed to insure $0 \in G$. Suppose by way of contradiction that F is a finite nonempty subset of $\mathbb{N}$ and $F \in \langle G \rangle$. Let n denote the maximum element of F. then $F - n = \{0\} \in \langle G \rangle$, contradicting the hypothesis that $\langle G \rangle$ contains no singletons. Thus every nonempty element of $\langle G \rangle$ is infinite.

Let $X_0 = G$ and $x_0 = 0$. Given a nonempty set X_n in $\langle G \rangle$ and x_n, let x_{n+1} be the least element of X_n greater than x_n. Since X_n is a nonempty set in $\langle G \rangle$, it is infinite, so x_{n+1} must exist. Let $X_{n+1} = X_n \cap (X_n - x_{n+1})$. Note that if X_n contains 0, then so does X_{n+1}. Consequently, for every n, X_{n+1} is a nonempty element of $\langle G \rangle$.

Repeating the argument from the proof of Theorem 3, $\mathsf{FS}(\langle x_n \rangle_{n>0}) \subseteq X_0 = G$, so Hindman's Theorem holds for G. $\square$

Lemma 5. ($\mathsf{RCA}_0 + \Sigma^0_2$ induction) *If the downward translation algebra $\langle G \rangle$ contains finitely many singletons, then Hindman's Theorem holds for G.*

Proof. Suppose that $\langle G \rangle$ contains finitely many singletons. We will show that there is a set H differing from G only on a finite initial interval, such that $\langle H \rangle$ contains no singletons.

If $\langle G \rangle$ contains no singletons, let $H = G$. Otherwise, let $\{m\}$ be the largest singleton in $\langle G \rangle$. Let $G_0, \ldots G_{2^{m+1}-1}$ be an enumeration of all subsets of $\mathbb{N}$ differing from G only at or below m. Note that by applying translations and Boolean operations to G and $\{m\}$, we may construct each G_i. Thus for each i, $\langle G_i \rangle \subseteq \langle G \rangle$ and in particular, the singletons of each $\langle G_i \rangle$ are a subset of the singletons of $\langle G \rangle$. Furthermore, the assertion "$\{j\}$ is an element of $\langle G_i \rangle$" holds if and only if there is a finite collection of translations of G_i and G_i^c such that for every n, n is in the intersection of the translations if and only if $n = j$. This assertion is expressible by a Σ^2_0 formula. By bounded Σ^2_0 comprehension (which is provable in RCA_0 from Σ^2_0 induction [5]), there is a set X such that for every $i < 2^{m+1} - 1$ and $j \leq m$, $(i, j) \in X$ if and only if $\{j\} \in G_i$. If there is an i such that G_i contains no singletons, pick the first such i and let $H = G_i$. Otherwise use X to define the set $Y = \{m_i \mid i < 2^{m+1} - 1\}$, the collection containing the maximum singleton from each G_i. Let p be the minimum of Y, and let H be the first G_i for which p is the largest singleton.

We will now show that $\langle H \rangle$ contains no singletons. If $\langle H \rangle$ contains a singleton, then it contains $\{p\}$ as constructed above. Since $\langle H \rangle = \langle H^c \rangle$ we may relabel as needed to insure $p \in H$.

Because $\langle H \rangle$ is a boolean algebra generated by downward translations of H and H^c, $\{p\}$ is expressible as a union of intersections of translations of H and H^c. Furthermore, because $\{p\}$ is a singleton, we may discard all but the first element of the union, and write $\{p\}$ as an intersection of translations of H and H^c. By way of contradiction, suppose $\{p\}$ is expressed as the intersection of a collection of such translations, and H itself is not in the collection. Since $p \notin H^c$, the collection must consist entirely of non-zero translations of H and H^c. Adding the value of the smallest translation to each of these sets yields a collection of elements of $\langle H \rangle$ whose intersection is a singleton which is greater than p, contradicting the fact that $\{p\}$ is the largest singleton in $\langle H \rangle$. Thus, H must be included in any intersection defining $\{p\}$, and we may write

$$\{p\} = H \cap \bigcap_{k \in F} (H^{d_k} - k)$$

where F is a finite set of positive natural numbers and d_k indicates whether or not to take the complement of H. Note that $\cap_{k \in F}(H^{d_k} - k) \subseteq H^c \cup \{p\}$.

Now consider $\langle H' \rangle$ where $H' = H - \{p\}$. Suppose by way of contradiction that $\{p\} \in \langle H' \rangle$. As argued above, we must be able to express $\{p\}$ as the intersection of a collection of translations of H' and H'^c. Since $p \notin H'$, we know H' is not in the collection. If H'^c is not in the collection, then $\langle H' \rangle$ contains a singleton larger than p. However, $H' = H \cap \{p\}^c \in \langle H \rangle$, so $\langle H' \rangle \subseteq \langle H \rangle$, implying that $\langle H' \rangle$ can contain no singletons larger than p. Thus any intersection defining $\{p\}$ in $\langle H' \rangle$ includes H'^c, and we may write

$$\{p\} = H'^c \cap \bigcap_{k \in W} (H'^{d_k} - k)$$

where W is a finite collection of positive integers. Note that $\cap_{k \in W}(H'^{d_k} - k) \subseteq (H'^c)^c \cup \{p\} = H$.

Combining the preceding two centered equations, we have

$$\{p\} \subseteq \bigcap_{k \in F} (H^{d_k} - k) \cap \bigcap_{k \in W} (H'^{d_k} - k) \subseteq (H^c \cup \{p\}) \cap H = \{p\}.$$

Since $H' \in \langle G \rangle$, this shows that $\{p\}$ is expressible as an intersection consisting entirely of nonzero translations of H and H^c, yielding a contradiction. Thus, $\{p\} \notin \langle H' \rangle$. Since $\langle H' \rangle$ is closed under downward translation, this shows that any singletons in $\langle H' \rangle$ must be strictly less than p. However, H' differs from G only at or below $m \geq p$, so H' must contain a singleton greater than or equal to p, yielding a final contradiction and proving that $\langle H \rangle$ contains no singletons.

We have shown that H differs from G only at or below m, and H has no singletons. By Lemma 4, Hindman's Theorem holds for H. Given an infinite homogeneous set Z for H, the set $\{n \in Z \mid n > m\}$ is an infinite homogeneous set for G. Thus Hindman's Theorem holds for G.

□

Theorem 6. ($\mathsf{RCA}_0 + \Sigma^0_2$ induction) *If the downward translation algebra $\langle G \rangle$ doesn't contain all the singletons, then Hindman's Theorem holds for G.*

Proof. By closure under downward translation, if $\langle G \rangle$ contains a singleton $\{p\}$, it contains all singletons $\{n\}$ such that $n < p$. Thus, if $\langle G \rangle$ doesn't contain all the singletons, it must contain at most finitely many singletons. By Lemma 5, Hindman's Theorem holds for G. □

4 A proof of Stone's Theorem

A (code for a) countable Boolean algebra consists of a set of elements A (including 0 and 1 as designated elements) and operations $\cap$, $\cup$, and ^{c} satisfying the usual axioms. (For a list of the usual axioms see page 5 of [1].)

Theorem 7 (Stone's Representation Theorem). (ACA_0) *Every countable Boolean algebra is isomorphic to a countable field of sets.*

Proof sketch. Given a Boolean algebra A, form the 0–1 tree T of finite initial segments of characteristic functions for ultrafilters on A. The set of infinite paths through T is a closed subset of Cantor space. Applying Brown's result (Theorem 3.3 of [2]), select a countable dense set of paths, $\{P_i \mid i \in \mathbb{N}\}$. To each element $a \in A$, assign the set $X_a = \{n \mid a \in P_n\}$. The collection $\{X_a \mid A \in A\}$ and the function $f : a \to \{X_a \mid A \in A\}$ defined by $f(a) = X_a$ both exist by arithmetical comprehension. The verification that f is an isomorphism is straightforward. □

Corollary 8. *If A is an arithmetical countable Boolean algebra, then there is an arithmetically definable sequence B of subsets of $\mathbb{N}$ which is a countable Boolean algebra, and there is an arithmetical Boolean algebra isomorphism between A and B.*

References

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