Leaf Management

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Leaf sets

A leaf in a tree is a node with no extensions.

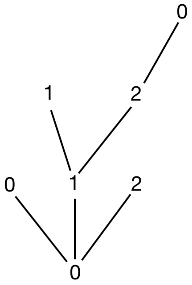
Given a computable subtree T of $\mathbb{N}^{\mathbb{N}}$, we have two situations:

- If there is function f such that $f(n)$ is a bound on the node labels at level n, then we can compute the set of leaves of T.
- In the absence of such a function, the leaf set may not be computable.

For computable subtrees of $2^{\mathbb{N}}$, we can always compute a leaf set. On the other hand, ω branching trees often present difficulties.

A tree transformation

Here is a recipe for transforming a tree with a potentially uncomputable leaf set to one with a leaf set computable from the tree.



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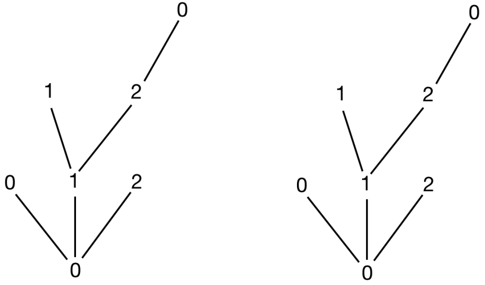

Start with a copy of the tree.

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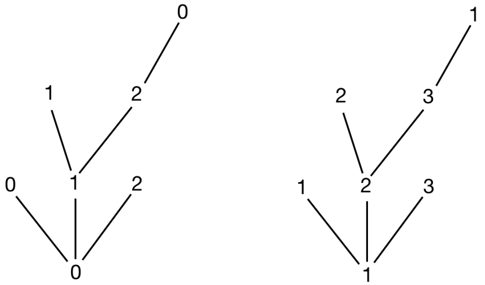

Add 1 to each node.

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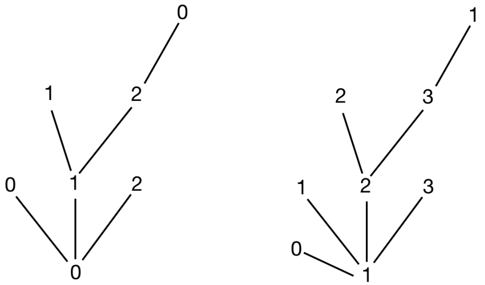

Extend the root node by concatenating with 0.

A tree transformation

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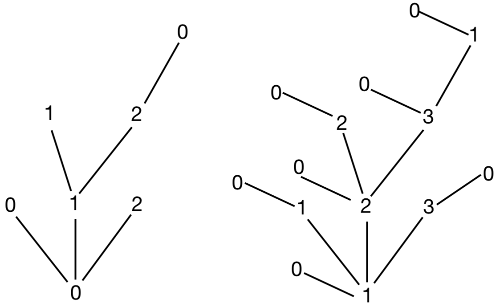
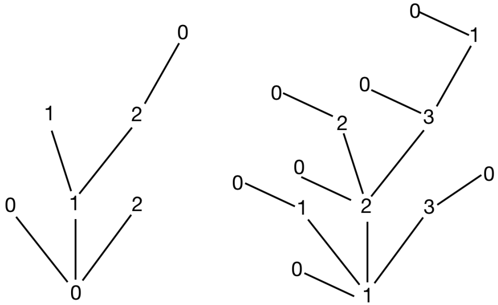


Repeat with the remaining nodes.

The transformed tree

A sequence is a leaf in the transformed tree iff it is $\tau^\frown 0$ for some τ in the original tree.

The new tree is well-founded iff the original tree is. It has a unique infinite path iff the original tree does.



A reverse math consequence

Thm: (RCA_0) The follow are equivalent.

1. Π^1_1-CA_0.
2. If $\langle T_i \rangle_{i \in \mathbb{N}}$ is a sequence of trees in $\mathbb{N}^{<\mathbb{N}}$, then there is a function $f : \mathbb{N} \to 2$ such that $f(i) = 1$ if and only if T_i contains an infinite path. (Lemma VI.1.1 of Simpson [2])
3. If $\langle T_i \rangle_{i \in \mathbb{N}}$ is a sequence of trees and $\langle L_i \rangle_{i \in \mathbb{N}}$ is a sequence of sets such that for each i, L_i is the set of leaves of T_i, then there is a function $f : \mathbb{N} \to 2$ such that $f(i) = 1$ if and only if T_i contains an infinite path.

Joint work with C. Davis and J. Pardo.

A theorem on hypergraphs

Hypergraphs generalize graphs by allowing more than two vertices in an edge.
A proper coloring of a hypergraph is a function that is not constant on any edge.

Thm: (RCA_0) For each $k \geqslant 2$, the following are equivalent.

1. Π^1_1-CA_0.
2. If $\langle H_i \rangle_{i \in \mathbb{N}}$ is a sequence of hypergraphs, then there is a function $f : \mathbb{N} \to 2$ such that $f(i) = 1$ if and only if H_i has a proper k-coloring.

In the reversal, we construct hypergraphs corresponding to trees. By using trees with leaf sets, we were able to avoid an initial proof of ACA_0 from item 2 (i.e. bootstrapping).

Joint work with C. Davis and J. Pardo.

Other uses

It should be easy to formulate “foliated” versions of theorems of reverse mathematics.

Thm: (RCA_0) The follow are equivalent.

1. ATR_0.
2. If $\langle T_i \rangle_{i \in \mathbb{N}}$ is a sequence of trees in $\mathbb{N}^{<\mathbb{N}}$ each with at most one infinite path, then there is a function $f : \mathbb{N} \to 2$ such that $f(i) = 1$ if and only if T_i contains an infinite path. (Lemma V.5.2 of Simpson [2])

Similary, we could formulate “foliated” versions of many principles (e.g. Σ^1_1-CA^-) and show that they are Weihrauch equivalent to the usual forms. This may help in finding lower bounds for the Weihrauch strength of combinatorial principles.

References

[1] Caleb Davis, Jeffry Hirst, Jake Pardo, and Tim Ransom, *Reverse mathematics and colorings of hypergraphs* (2018), 1-13. Submitted. arXiv:1804.09638.

[2] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, 2009. DOI 10.1017/CBO9780511581007. MR2517689