Reverse Mathematics and Constructive Analysis

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The basic notion:

If certain types of statements are provable constructively, then they are uniformly computably provable.

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- A statement: $\forall X \exists Y \, \theta(X, Y)$
- Its *uniformization*: $\forall \langle X_n \mid n \in \mathbb{N} \rangle \exists \langle Y_n \mid n \in \mathbb{N} \rangle \forall n \, \theta(X_n, Y_n)$

Uniformity:

Some statements that are computably provable are not uniformly computably provable.

An axiomatization for computable analysis:

RCA: A subsystem of second order arithmetic including

classical predicate calculus

ordered semi-ring axioms

induction

Δ^0_1-comprehension

There is a infinite computable $0 - 1$ tree with no computable infinite path, so RCA does not prove Weak König's Lemma (WKL).

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An example: Suppose we encode real numbers using rapidly converging sequences of rationals.

Theorem

(RCA) *If* $X = \langle x_0, x_1, \ldots, x_n \rangle$ *is a finite sequence of real numbers, then there is a* $j \leqslant n$ *such that* x_j *is the minimum of* X.

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Theorem

(RCA) *The following are equivalent:*

1. WKL.
2. *If* $\langle X_n \mid n \in \mathbb{N} \rangle$ *is an infinite sequence of finite sequences of real numbers, then there is a sequence* $\langle j_n \mid n \in \mathbb{N} \rangle$ *of natural numbers such that for all* n*, the minimum of* X_n *is* x_{j_n}.

The relation to constructive analysis

An axiomatization for a fragment of constructive analysis

$\text{E-HA}^{\omega} + \text{AC}$:

Intuitionistic arithmetic in all finite types including

intuitionistic predicate calculus

induction

primitive recursion (on all finite types)

an **E**xtensionality scheme: $x = y \to z(x) = z(y)$

Axiom of **C**hoice: $\forall x \exists y A(x, y) \to \exists Y \forall x A(x, Y(x))$

Extending RCA to all finite types

The axiom system RCA^{ω} is a conservative extension of RCA to all finite types. (Any formula in the language of RCA which is provable in RCA^{ω} is also provable in RCA.)

RCA^{ω} can be axiomatized as

$$\text{E-HA}^{\omega} + \mathrm{QF} - \mathrm{AC}^{1,0} + \text{excluded middle}$$

The main result

Theorem

Suppose $\theta(X, Y)$ *is in* Γ_1. *(More about this soon.)*

If

$$\text{E-HA}^\omega + \text{AC} \vdash \forall X \exists Y\, \theta(X, Y)$$

then

$$\text{RCA}^\omega \vdash \forall \langle X_n \mid n \in \mathbb{N} \rangle \exists \langle Y_n \mid n \in \mathbb{N} \rangle \forall n\, \theta(X_n, Y_n).$$

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The role of Γ_1: If $\theta \in \Gamma_1$, then $\text{E-HA}^\omega \vdash (t \,\text{mr}\, \theta) \to \theta$. That is, E-HA^ω proves that if θ is modified realizable, then θ holds.

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Note: If θ is in the language of RCA, then RCA^ω may be replaced by RCA in the theorem.

An application

The contrapositive of the theorem states that if $\theta(X, Y)$ is (a formula in the language of RCA) in Γ_1 and

$$\mathrm{RCA} \nvdash \forall \langle X_n \mid n \in \mathbb{N} \rangle \exists \langle Y_n \mid n \in \mathbb{N} \rangle \forall n\, \theta(X_n, Y_n)$$

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We know

RCA $\nvdash$ For every infinite sequence of finite sequences of reals we can find a sequence of indices of their minima.

so E-HA$^{\omega}$ + AC $\nvdash$ For every finite sequence of reals, we can find the index of its minimum.

Variations

For $\theta \in \Gamma_1$,

If $\text{E-HA}^\omega + \text{AC} + \text{IP}^\omega_{\text{ef}} \vdash \forall X \, \exists Y \, \theta(X, Y)$ then
$\text{RCA}^\omega \vdash \forall \langle X_n \mid n \in \mathbb{N} \rangle \, \exists \langle Y_n \mid n \in \mathbb{N} \rangle \, \forall n \, \theta(X_n, Y_n)$.

If $\widehat{\text{E-HA}}^\omega_{\restriction} + \text{AC} + \text{IP}^\omega_{\text{ef}} \vdash \forall X \exists Y \, \theta(X, Y)$ then
$\text{RCA}^\omega_0 \vdash \forall \langle X_n \mid n \in \mathbb{N} \rangle \, \exists \langle Y_n \mid n \in \mathbb{N} \rangle \, \forall n \, \theta(X_n, Y_n)$.

For $\theta \in \Gamma_2$ (using the *Dialectica* interpretation),

If $\text{WE-HA}^\omega + \text{AC} + \text{IP}^\omega_{\forall} + \text{M}^\omega \vdash \forall X \exists Y \, \theta(X, Y)$ then
$\text{RCA}^\omega \vdash \forall \langle X_n \mid n \in \mathbb{N} \rangle \, \exists \langle Y_n \mid n \in \mathbb{N} \rangle \forall n \, \theta(X_n, Y_n)$.

If $\widehat{\text{WE-HA}}^\omega_{\restriction} + \text{AC} + \text{IP}^\omega_{\forall} + \text{M}^\omega \vdash \forall X \exists Y \, \theta(X, Y)$ then
$\text{RCA}^\omega_0 \vdash \forall \langle X_n \mid n \in \mathbb{N} \rangle \, \exists \langle Y_n \mid n \in \mathbb{N} \rangle \forall n \, \theta(X_n, Y_n)$.

Questions

1. Can the families Γ_1 and Γ_2 in the theorems be expanded to larger nicely characterized families?

2. In applying the contrapositive, do the reversals provide additional useful information about the nature of the nonconstructivity of the initial statement?

3. Could a computable restriction of the uniformized statement assist in discovering a constructive restriction of the initial statement?

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