Graphs, reverse mathematics, and Weihrauch reductions

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Reverse mathematics: The method

Reverse Mathematics measures the strength of theorems by proving equivalence results over. . .

The base theory RCA_0

Reverse mathematics: The method

Reverse Mathematics measures the strength of theorems by proving equivalence results over...

The base theory RCA_0:

- Variables for natural numbers and sets of natural numbers
- Axioms
  - Arithmetic axioms
    (e.g. $n + 0 = n$ and $n + (m') = (n + m)'$)
  - Induction for particularly simple formulas
  - Recursive comprehension:
    If you can compute a set, then it exists.

A theorem of RCA_0

Thm: (RCA_0) For any $X \subset \mathbb{N}$, the set $Y = \{n+1 \mid n \in X\}$ exists.

An example:

n	0	1	2	3	4	5	6
χ_x	1	0	0	1	1	0	0
χ_y	0	1	0	0	1	1	0

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An example:

n	0	1	2	3	4	5	6
χ_X	1	0	0	1	1	0	0
χ_Y	0	1	0	0	1	1	0

A proof sketch: Given χ_X, define

$$\chi_Y(n) = \begin{cases} 0 & \text{if } n = 0, \\ \chi_X(n-1) & \text{if } n \neq 0. \end{cases}$$

An equivalence theorem!

Thm: (RCA_0) The following are equivalent:

(1) WKL_0: Every infinite 0-1 tree has an infinite path.

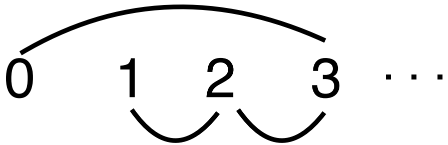
(2) If every finite subgraph of G can be 2-colored, then G can be 2-colored.

Proof sketch:

(1)$\rightarrow$(2) Given a graph, build a tree such that every path computes a coloring.

(2)$\rightarrow$(1) Given a tree, build a graph such that every 2-coloring computes a path.

Graph→tree and path→coloring



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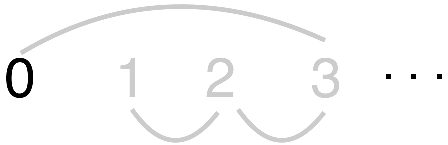

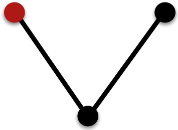

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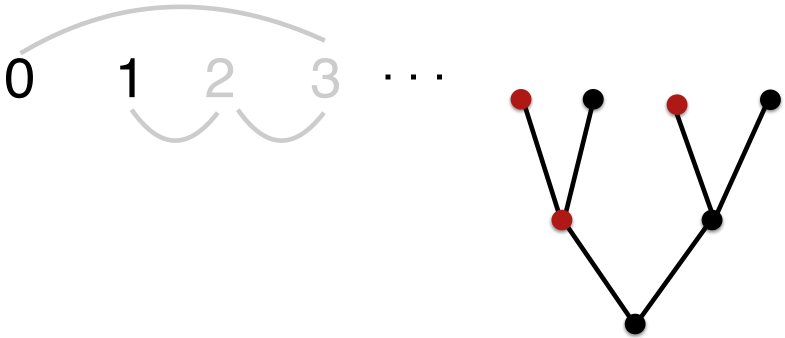

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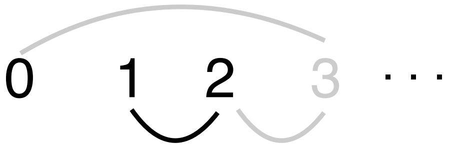

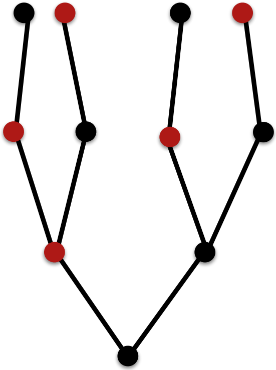

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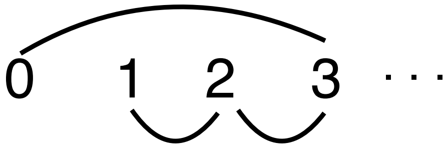

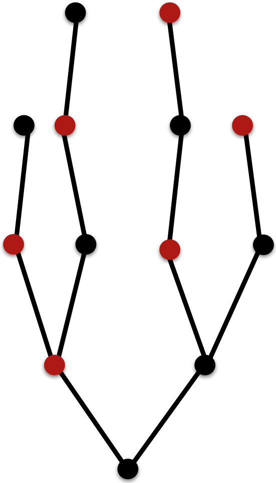

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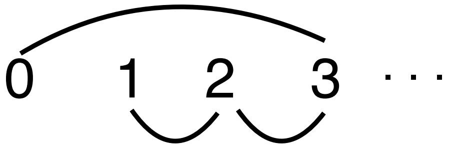

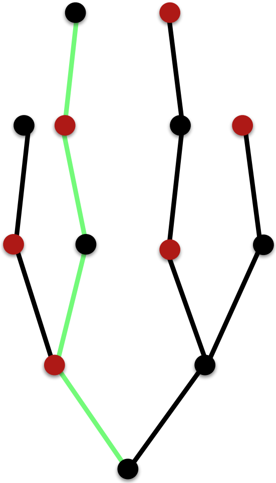

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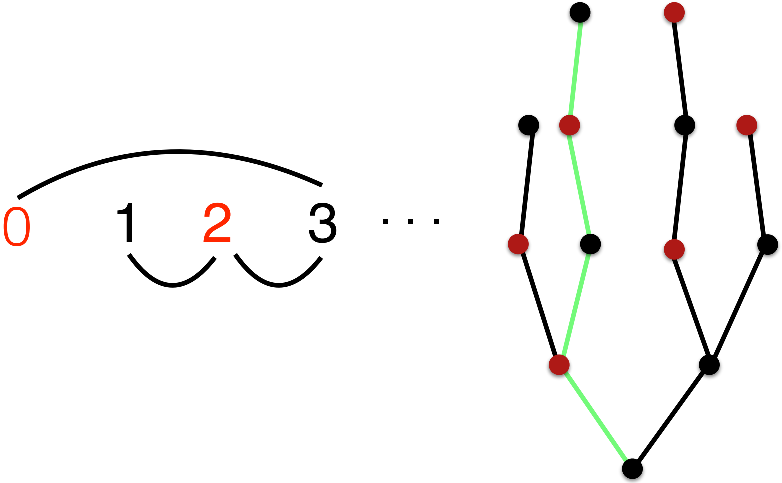
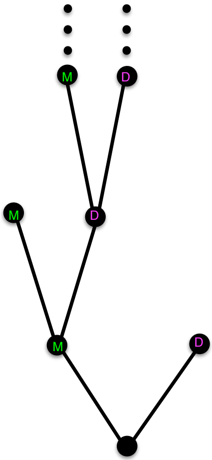


Tree→graph and coloring→path



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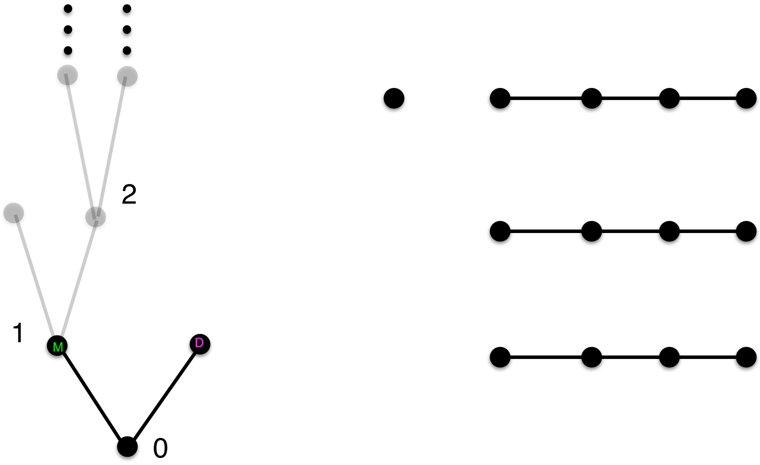

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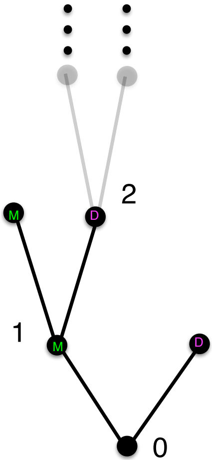

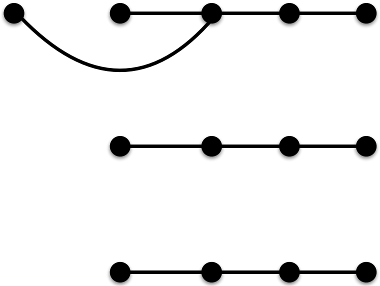

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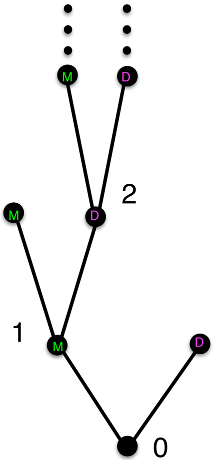

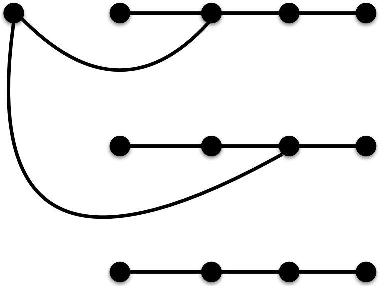

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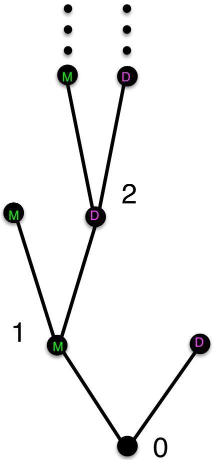

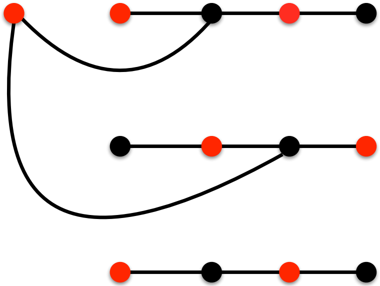

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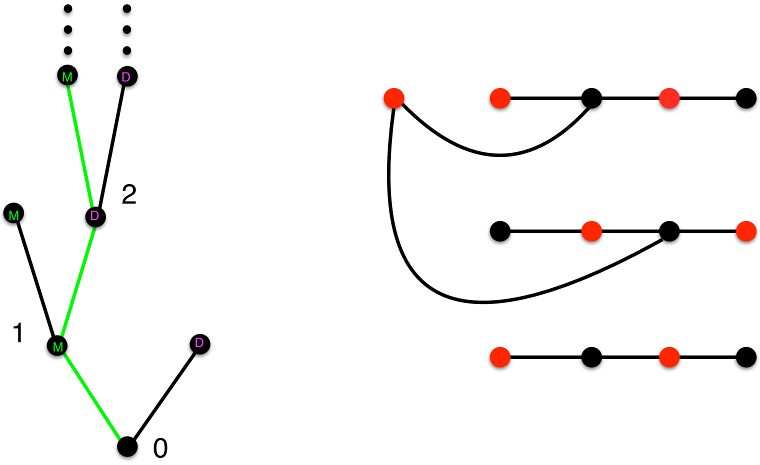

By the way: More things equivalent to WKL_0

Thm: (RCA_0) The following are equivalent:

1. WKL_0
2. If every finite subgraph of G is 2-colorable, then G is 2-colorable. [4]
3. Every continuous function on $[0, 1]$ is bounded. [9]
4. Every continuous function on $[0, 1]$ is Riemann integrable. [9] [7]
5. Every open cover of $[0, 1]$ has a finite subcover. [2]
6. Every countable commutative ring has a prime ideal. [3]

Many theorems of mathematics are either provable in RCA_0 or equivalent to one of: WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0

An alternative approach: Weihrauch reductions

We consider *problems* of the form $\mathrm{P} : \forall x(p_1(x) \to \exists y\, p_2(x,y))$

$p_1(x)$	means x is an instance of the problem P
$p_2(x,y)$	means y is a solution of the instance x of the problem P.

In this setting $\mathrm{Q} \leqslant_W \mathrm{P}$ means there are computable functionals ψ and φ such that

$$\begin{array}{ccc} & \psi & \\ x_Q & \longrightarrow & x_P \\ \downarrow & & \downarrow \\ y_Q & \longleftarrow & y_P \\ & \varphi & \end{array}$$

Note: φ can use information about x_Q to compute y_Q. (This is weak reduction.)

Weihrauch reductions: A concrete example

Consider the problems:

Problem P: Every infinite 0-1 tree has a path.

Problem Q: Every locally 2-colorable graph has a 2-coloring.

Our previous argument actually shows Q $\leqslant_W$ P:

$$\begin{array}{ccc} & \psi & \\ x_Q & \longrightarrow & x_P \\ \downarrow & & \downarrow \\ y_Q & \longleftarrow & y_P \\ & \varphi & \end{array}$$

where

ψ turns graphs into trees, and φ turns paths into colorings.

We also showed $P \leqslant_W Q$, so $P \equiv_W Q$.

Weihrauch reduction: Extra milage

In the Weihrauch literature, $\widehat{P}$ is used to denote the parallelization of P. A problem for $\widehat{P}$ consists of an infinite sequence of instances of P, and a solution for $\widehat{P}$ consists of the associated infinite sequence of solutions. Because of the uniformity of Weihrauch reductions, we have the following general result.

Thm: If $P \leqslant_W Q$ then $\widehat{P} \leqslant_W \widehat{Q}$.

Consequently, if

P: Every infinite 0-1 tree has a path.

Q: Every locally 2-colorable graph has a 2-coloring.

then $\widehat{P} \equiv_W \widehat{Q}$.

Toward formalizing Weihrauch reductions

We would like to get proof theoretic results from Weihrauch reductions. The functionals φ and ψ are not objects of second order arithmetic. However, an analog of RCA_0 exists for higher order objects.

An axiom system formulated by Kohlenbach [6]

RCA_0^ω includes:

$\widehat{\text{E-HA}}^\omega_\restriction$	Formal arithmetic in all finite types with restricted induction and primitive recursion
	The law of the excluded middle ($A \vee \neg A$)
$\text{QF-AC}^{1,0}$	A choice scheme that implies the recursive comprehension axiom (RCA)

Formalizing Weihrauch reductions

Given problems:
P : $\forall x(p_1(x) \to \exists y\, p_2(x, y))$ and Q : $\forall x(q_1(x) \to \exists y\, q_2(x, y))$

in the language of RCA_0^ω, we use $\mathrm{Q} \leqslant_W \mathrm{P}$ to abbreviate

$$\exists\varphi\exists\psi\forall u\,(q_1(u) \to (p_1(\varphi(u)) \wedge \forall y[p_2(\varphi(u), y) \to q_2(u, \psi(u, y))]))$$

Which says that there are functionals φ and ψ such that

$q_1(u)$	If u is an instance of Q
$p_1(\varphi(u))$	then $\varphi(u)$ is an instance of P
$p_2(\varphi(u), y)$	such that whenever y is a solution the instance $\varphi(u)$ of the problem P
$q_2(u, \psi(u, y))$	$\psi(u, y)$ computes a solution to the instance u of the problem Q

For many problems, if $i\mathrm{RCA}_0^\omega$ proves that such a φ and ψ exist, then analogous computable functionals exist. (The converse is not true and the use of the intuitionistic system matters.)

Formalized reductions

Useful features of formalized Weihrauch reductions:

Thm: If P and Q are nice and $i\mathrm{RCA}_0^\omega \vdash P \leqslant_W Q$, then $P \leqslant_W Q$.

For more about $i\mathrm{RCA}_0^\omega$ and *nice* see Hirst and Mummert's "Using Ramsey's Theorem Once."

Thm: If $\mathrm{RCA}_0^\omega \vdash P \leqslant_W Q$, then $\mathrm{RCA}_0 \vdash Q \to P$.

Thm: $\mathrm{RCA}_0^\omega \vdash P \leqslant_W Q \to \widehat{P} \leqslant_W \widehat{Q}$.

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Thm: $\mathrm{RCA}_0^\omega \vdash P \leqslant_W Q \to \widehat{P} \leqslant_W \widehat{Q}$.

If we write

WKL_0 for "every infinite 0-1 tree has an infinite path"

G for "every locally 2-colorable graph has a 2-coloring"

then

$\mathrm{RCA}_0^\omega \vdash \mathrm{WKL}_0 \equiv_W G$

$\mathrm{RCA}_0^\omega \vdash \widehat{\mathrm{WKL}_0} \equiv_W \widehat{G}$

$\mathrm{RCA}_0 \vdash \widehat{\mathrm{WKL}_0} \leftrightarrow \widehat{G}$

More reverse math consequences

We know that: $\mathrm{RCA}_0 \vdash \widehat{\mathrm{WKL}_0} \leftrightarrow \widehat{G}$

It is also known that: $\mathrm{RCA}_0 \vdash \widehat{\mathrm{WKL}_0} \leftrightarrow \mathrm{WKL}_0$

Thm: (RCA_0) The following are equivalent:

1. WKL_0
2. $\widehat{\mathrm{WKL}_0}$
3. G: Every locally 2-colorable graph is 2-colorable.
4. $\widehat{G}$: Every infinite sequence of locally 2-colorable graphs has a corresponding infinite sequence of 2-colorings.

Note: RCA_0^ω proves that $G \equiv_W \mathrm{WKL}_0 \equiv_W \widehat{G} \equiv_W \widehat{\mathrm{WKL}_0}$.

Parallelization can affect strength

For the 2-coloring problem G, $G \equiv_W \widehat{G}$ and $\mathrm{RCA}_0 \vdash G \leftrightarrow \widehat{G}$.

Not all combinatorial theorems behave like G when parallelized. For example...

Ramsey's Theorem for pairs and two colors $\mathrm{RT}(2,2)$ says: If the edges of an infinite complete graph are 2-colored, then there is an infinite subset of the vertices such that the corresponding complete subgraph is monochromatic.

$\mathrm{RCA}_0 \vdash \widehat{\mathrm{RT}(2,2)} \leftrightarrow \mathrm{ACA}_0$, but by Seetapun and Slaman's theorem [8], $\mathrm{RCA}_0 \nvdash \mathrm{RT}(2,2) \rightarrow \mathrm{ACA}_0$.

Parallelization can affect strength

For Weihrauch reducibility, just two applications of Ramsey's theorem cannot be reduced to a single use,

$$(\mathrm{RT}(2,2), \mathrm{RT}(2,2)) \not\leqslant_W \mathrm{RT}(2,2)$$

This is a consequence of the *Squashing Theorem* of Dzhafarov et al [1], which shows that if true,

$$(\mathrm{RT}(2,2), \mathrm{RT}(2,2)) \leqslant_W \mathrm{RT}(2,2)$$

would imply

$$\widehat{\mathrm{RT}(2,2)} \leqslant_W \mathrm{RT}(2,2)$$

contradicting a theorem of Jockusch [5]

Of course, $\mathrm{RCA}_0 \vdash \mathrm{RT}(2,2) \to (\mathrm{RT}(2,2), \mathrm{RT}(2,2))$, so the connection between provability and Weihrauch reducibility is not simple. (Advertisement for the Logic Colloquium.)

A word on the Squashing theorem:

An idea from the proof of the:

Squashing Theorem: $\langle P, P \rangle \leqslant_W P$ *implies* $\widehat{P} \leqslant_W P$

(provided P is nice)

Compress the sequence $f0, f1, \ldots$ into a single instance $h0$.

$$h0 \begin{cases} f0 & \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ f1 & \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \end{cases}$$

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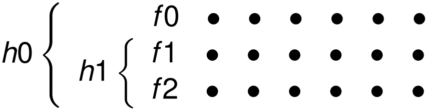

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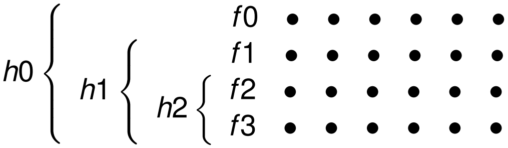

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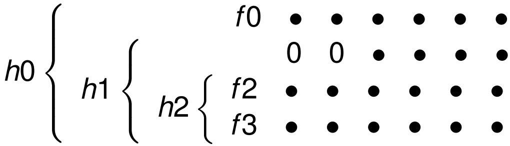
Assume the initial outputs of $h1$ are 0.



A word on the Squashing theorem:

An idea from the proof of the:

Squashing Theorem: $\langle P, P \rangle \leqslant_W P$ *implies* $\widehat{P} \leqslant_W P$
(provided tails of solutions of P are solutions)

Assume the initial outputs of $h2$ are 0.

$$
h0 \left\{ \begin{array}{l} \qquad\qquad\qquad f0 \\ h1 \left\{ \begin{array}{l} \qquad\qquad \\ h2 \left\{ \begin{array}{l} \\ f3 \end{array} \right. \end{array} \right. \end{array} \right.
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
0 & 0 & \bullet & \bullet & \bullet & \bullet \\
0 & 0 & 0 & 0 & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}
$$

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