Reverse Mathematics and Gödel's *Dialectica* Interpretation

Jeffry L. Hirst

Appalachian State University

JMM in San Diego

January 8, 2008

These slides are available at: `www.mathsci.appstate.edu/~jlh`

Gödel's Main *Dialectica* Result

Thm 1. *If* $\widehat{\mathsf{HA}}^{\omega}$ *proves a formula* θ, *then* RCA_0^{ω} *proves the related* $\exists\forall$ *formula* θ^D.

- $\widehat{\mathsf{HA}}^{\omega}$ is an axiom system for constructive analysis, with:

  intuitionistic predicate calculus (no law of the excluded middle), restricted induction, and axioms pertaining to objects of higher types.

- RCA_0^{ω} is an axiom system for computable analysis, with:

  classical logic, restricted induction and set comprehension, and axioms extending RCA_0 to objects of higher types.

Note: RCA_0^{ω} is $\mathsf{E\text{-}PRA}^{\omega} + \mathsf{QF} - \mathsf{AC}^{1,0}$.

Abbreviated definition of the *Dialectica* translation

(1) If φ is quantifier-free then $\varphi^D = \varphi_D = \varphi$.

If $\varphi^D = \exists x \forall y \varphi_D$ and $\psi^D = \exists u \forall v \psi_D$, translate more complicated formulas as follows:

(2) $(\varphi \wedge \psi)^D = \exists x \exists u \forall y \forall v \, (\varphi_D \wedge \psi_D)$.

(3) $(\varphi \vee \psi)^D = \exists z \exists x \exists u \forall y \forall v \, ((z = 0 \wedge \varphi_D) \vee (z = 1 \wedge \psi_D))$.

(4) $(\forall z \, \varphi(z))^D = \exists X \forall z \forall y \, \varphi_D(X(z), y, z)$.

(5) $(\exists z \, \varphi(z))^D = \exists z \exists x \forall y \, \varphi_D(x, y, z)$.

(6) $(\varphi \to \psi)^D = \exists U \exists Y \forall x \forall v \, (\varphi_D(x, Y(x, v)) \to \psi_D(U(x), v))$.

The negation $\neg\varphi$ is treated as an abbreviation of $\varphi \to \bot$.

An example

Suppose $\theta = \neg\forall y \exists x \forall z \neg(f(z) = y \wedge f(x) \neq y)$

$$\begin{aligned}
\theta^D &= (\neg\forall y \exists x \forall z \neg(f(z) = y \wedge f(x) \neq y))^D \\
&= (\neg\exists x^1 \forall y \forall z \neg(f(z) = y \wedge f(x(y)) \neq y))^D \\
&= (\exists x^1 \forall y \forall z \neg(f(z) = y \wedge f(x(y)) \neq y) \rightarrow \perp)^D \\
&= (\forall x^1 \exists y \exists z \neg\neg(f(z) = y \wedge f(x(y)) \neq y))^D \\
&= \exists y^{1\rightarrow 0} \exists z^{1\rightarrow 0} \forall x^1 \neg\neg(f(z(x)) = y(x) \wedge f(x(y(x)) \neq y(x))
\end{aligned}$$

Comment on type notation: 0 is the type of a natural number. $0 \rightarrow 0$ is the type of a function from natural numbers to natural numbers, and is often abbreviated by 1. $1 \rightarrow 0$ is the type of a functional that maps functions to numbers.

The connection between φ and φ^D

In a strong enough system, φ and φ^D are provably equivalent. (For example, $\widehat{\mathsf{HA}}^{\#}$, which consists of $\widehat{\mathsf{HA}}^{\omega}$ plus a strong choice scheme and some classical additions proves $\varphi \leftrightarrow \varphi^D$.)

The need for comprehension in one direction is clear.

Thm 2 (RCA_0^ω). *The scheme $\varphi \rightarrow \varphi^D$ implies ACA_0.*

Proof. For any function f, RCA_0^ω proves the formula (φ)

$$\forall y \exists x \forall z (f(z) = y \rightarrow f(x) = y).$$

φ^D is $\exists X^1 \forall y \forall z (f(z) = y \rightarrow f(X(y)) = y)$. If φ^D holds, then $\mathsf{Range}(f) = \{y \mid f(X(y)) = y\}$ exists. $\square$

The less obvious direction

Thm 3 (RCA_0^ω)**.** *The scheme* $\varphi^D \to \varphi$ *implies* ACA_0.

Outline of proof: Recall our first example of the *Dialectica* translation: Given $\theta = \neg\forall y \exists x \forall z \neg(f(z) = y \wedge f(x) \neq y)$, (which is equivalent to $\neg\forall y \exists x \forall z (f(z) = y \to f(x) = y)$), we have

$$\theta^D = \exists y^{1\to 0} \exists z^{1\to 0} \forall x^1 \neg\neg(f(z(x)) = y(x) \wedge f(x(y(x))) \neq y(x)).$$

Since RCA_0^ω proves $\neg\theta$, the scheme $\varphi^D \to \varphi$ implies $\neg(\theta^D)$. To finish the proof, use $\neg(\theta^D)$ to prove $\mathsf{Range}(f)$ exists.

Proof of Thm 3. continued

Suppose (for a contradiction) that for every function x of type 1, we can find a pair of integers (y, z) such that $(f(z) = y \wedge f(x(y)) \neq y)$. Apply $\mathsf{QF-AC}^{1,0}$ to find the function that picks the least pair, and then combine this with coordinate projections to get functions y and z of type $1 \longrightarrow 0$ such that

$$\forall x^1 (f(z(x)) = y(x) \wedge f(x(y(x))) \neq y(x)).$$

From this we can deduce θ^D, contradicting our assumption of $\neg(\theta^D)$.

Thus there is a function x of type 1 such that for every pair of integers y and z, we have $f(z) = y \longrightarrow f(x(y)) = y$.

$$\mathsf{Range}(f) = \{y \mid f(x(y)) = y\}.$$

Comparing *Dialectica* with Skolem Normal Form

If we write φ^P for the prenex form of φ, then $(\varphi^P)^D$ is the Skolem normal form of φ.

It's not hard to show that RCA_0^ω proves $(\varphi^P)^D \rightarrow \varphi$. Combined with the previous theorem, this yields:

Thm 4 (RCA_0^ω)**.** *The scheme* $\varphi^D \rightarrow (\varphi^P)^D$ *implies* ACA_0.

Conclusion: We can't uniformly computably convert the terms realizing the existential quantifiers in *Dialectica* translations into standard Skolem functions.

$\mathsf{Skolem} \to \textit{Dialectica?}$

Thm 5 (RCA_0^ω)**.** *The scheme* $(\varphi^P)^D \to \varphi^D$ *implies* WKL_0.

Idea of the proof: Let φ be the formula:

$$\forall y(\forall x(g_1(x) \neq y) \vee \forall w(g_2(w) \neq y))$$

asserting that g_1 and g_2 have disjoint ranges. RCA_0^ω proves that φ implies $(\varphi^P)^D$. However,

$$\varphi^D = \exists z^1 \forall y \forall x \forall w((z(y) = 0 \wedge g_1(x) \neq y) \vee (z(y) = 1 \wedge g_2(x) \neq y))$$

A separating set for the ranges of g_1 and g_2 can be derived from z.

Conclusion: We can't uniformly computably convert Skolem functions into *Dialectica* terms.

References

[1] Jeremy Avigad and Solomon Feferman. Gödel's functional ("Dialectica") interpretation. In *Handbook of proof theory*, volume 137 of *Stud. Logic Found. Math.*, pages 337–405. North-Holland, Amsterdam, 1998.

[2] Ulrich Kohlenbach. Higher order reverse mathematics. In *Reverse Mathematics 2001*, volume 21 of *Lecture Notes in Logic*, pages 281–295. ASL and A K Peters, 2005.

[3] A. S. Troelstra. Metamathematical Investigation of Intuitionistic Arithmetic and Analysis. Volume 344 of *Lecture Notes in Math.* Springer-Verlag, Berlin-Heidelberg-New York, 1973.