Reverse mathematics and colorings of hypergraphs

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College of Charleston

Reverse mathematics

Reverse mathematics uses a hierarchy of axioms of second order arithmetic to measure the strength of theorems.

The language has variables for natural numbers and sets of naturals numbers.

The base system, RCA_0, includes

- arithmetic facts (e.g. $n + 0 = n$),
- an induction scheme (restricted to Σ^0_1 formulas), and
- recursive comprehension
(computable sets exist, i.e. sets with programmable characteristic functions exist).

Adding stronger comprehension axioms creates stronger axiom systems.

ACA_0

The system ACA_0 adds arithmetical comprehension to RCA_0 (sets with arithmetically definable characteristic functions exist).

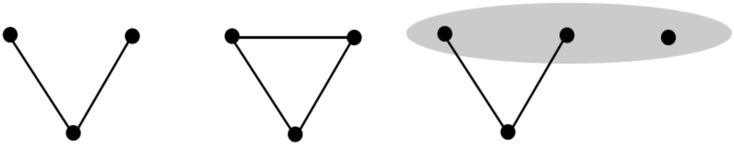
A theorem of reverse mathematics:

Theorem: Over RCA_0, the following are provably equivalent:

1. ACA_0.
2. Every injection has a range. (Lemma III.1.3, Simpson [5]).
3. Every countable sequence of reals in $[0, 1]$ has a convergent subsequence. (Friedman [3])

Proper colorings of hypergraphs

A *hypergraph* consists of vertices and edges. Edges may contain any number of vertices.



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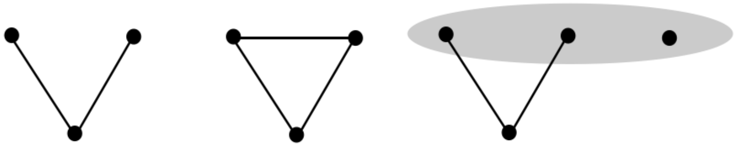
A 2-coloring is *proper* if no edge is monochromatic.



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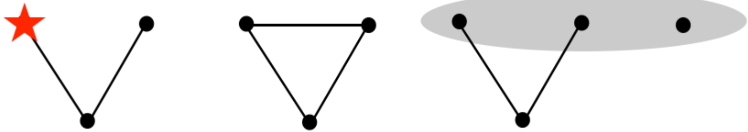

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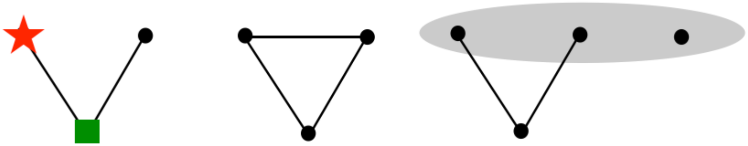

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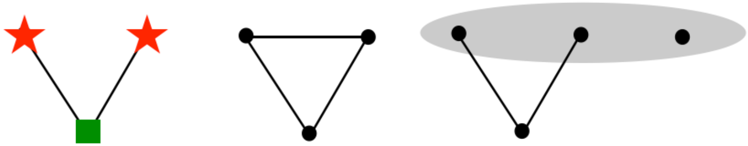

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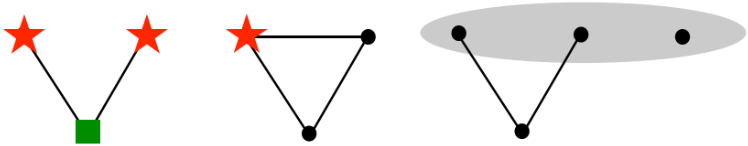

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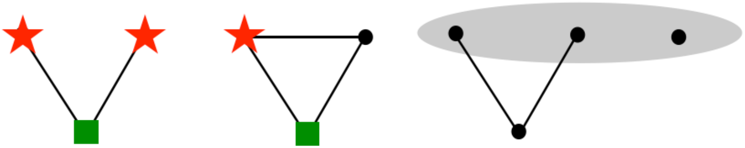

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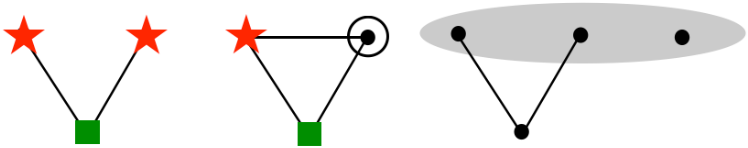

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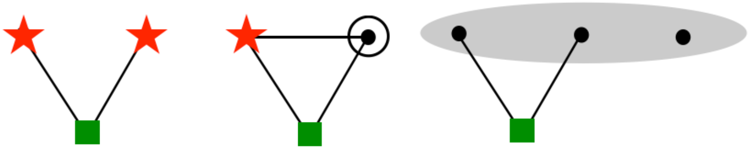

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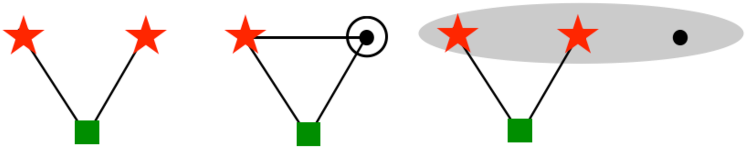

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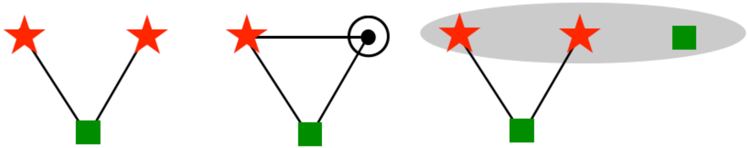

Hypergraphs with finite edges

Theorem: RCA_0 proves the following are equivalent:

(1) ACA_0.

(2) Suppose H is a hypergraph with finite edges presented as a sequence of characteristic functions. If every finite partial hypergraph of H has a proper 2-coloring, then H has a proper 2-coloring.

Proof sketch:

(1)$\rightarrow$(2): For every m, there is a least 2-coloring of $v_0, \ldots, v_m$ that can be extended to a proper 2-coloring of every finite partial hypergraph. Nesting these least 2-colorings yields a 2-coloring of all of H that is arithmetically definable (in H).

The reversal: Proper 2-colorings $\to$ ACA_0

Given an injection f, we want to build H so that the range of f can be computed from any 2-coloring of H.

For example, suppose $f(0) = 1$, $f(2) = 0$, and $2 \notin \mathrm{Range}(f)$.

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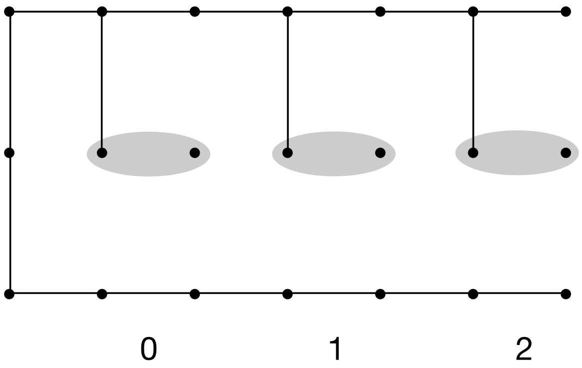

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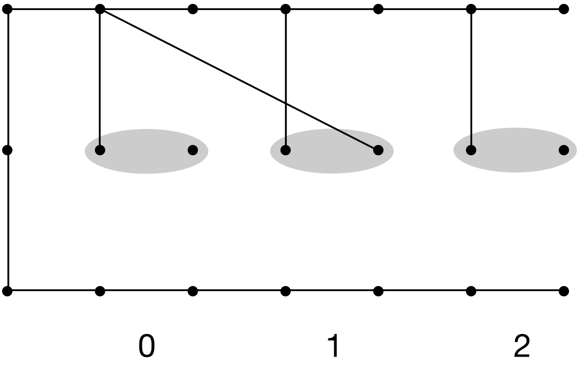

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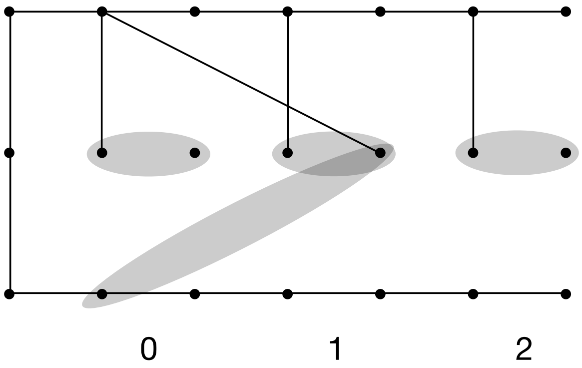

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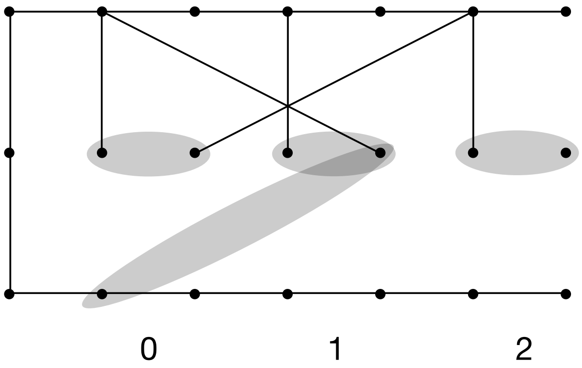

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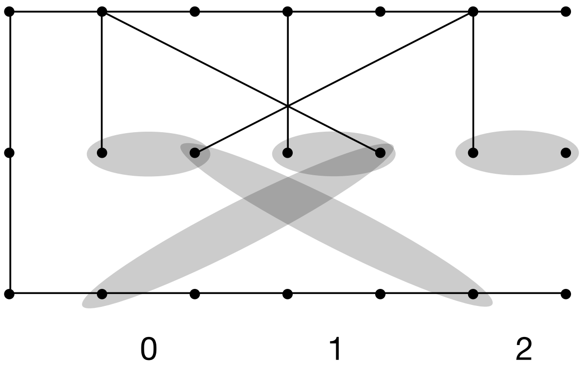

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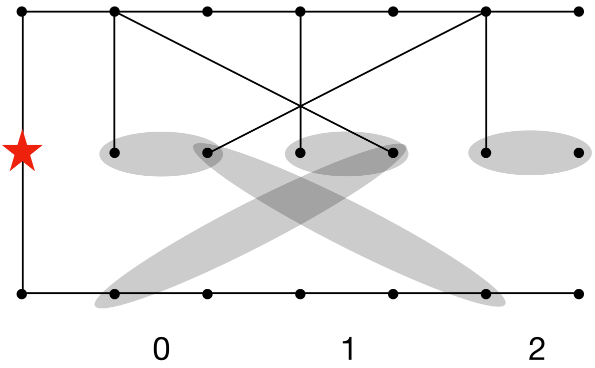

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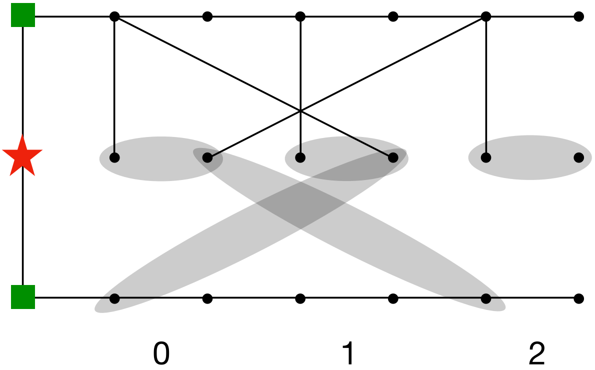

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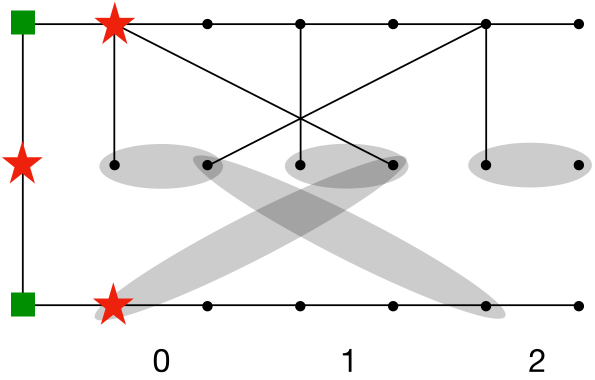

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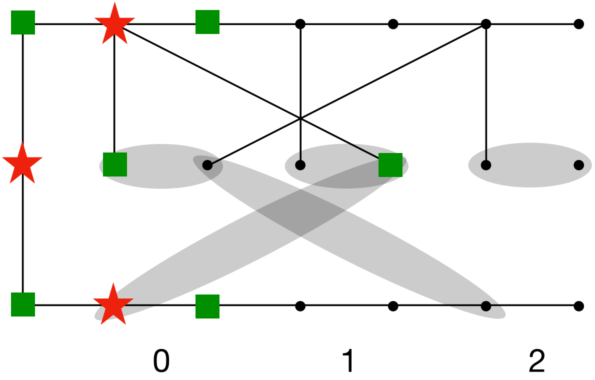

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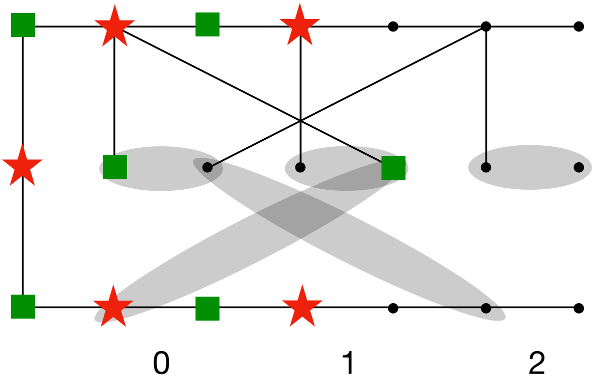

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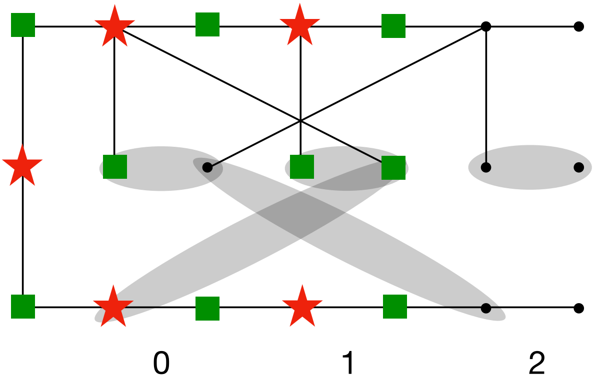

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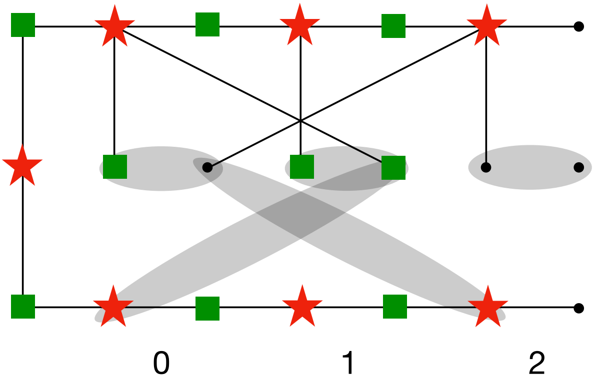

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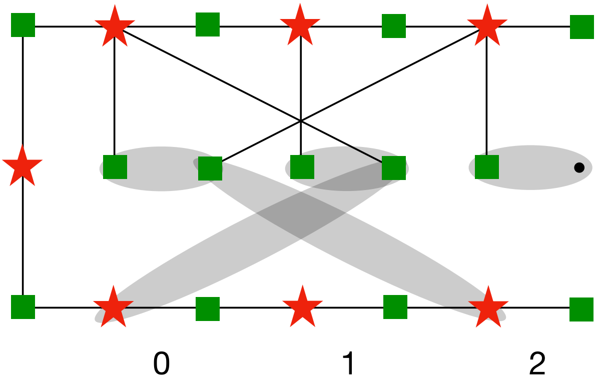

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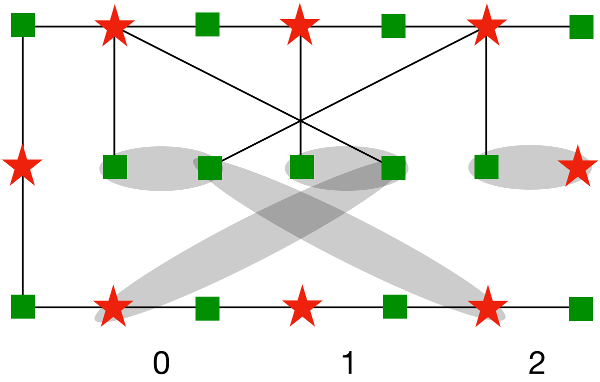


Hypergraphs with finite edges: Additional observations

Hypergraphs are different from graphs.

Theorem: RCA_0 proves the following are equivalent:

(1) ACA_0.

(2) Suppose H is a hypergraph with finite edges presented as a sequence of characteristic functions. If every finite partial hypergraph of H has a proper 2-coloring, then H has a proper 2-coloring.

Theorem: RCA_0 proves the following are equivalent:

(1) WKL_0.

(2) Suppose H is a graph with finite edges presented as a sequence of characteristic functions. If every finite partial graph of H has a proper 2-coloring, then H has a proper 2-coloring.

Hypergraphs with infinite edges

For hypergraphs with infinite edges, there is no arithmetical characterization of hypergraphs with proper 2-colorings. This is a corollary of:

Theorem: RCA_0 proves the following are equivalent:

(1) Π^1_1-CA_0, the comprehension scheme for Π^1_1 definable sets.

(2) $\widehat{\text{HC}}$: If $\langle H_i \rangle_{i \in \mathbb{N}}$ is a sequence of hypergraphs, then there is a function $f : \mathbb{N} \to 2$ such that $f(i) = 1$ if and only if H_i has a proper 2-coloring.

Proof sketch for $(1) \to (2)$:
$f(i) = 0$ if and only if every 2-coloring fails to be proper for H_i.
"Fails to be proper" means that for some j, all the vertices of edge E_j of H_i match.

Hypergraphs with infinite edges: the reversal

For the reversal, we need a combinatorial version of Π^1_1-CA_0.

Theorem: RCA_0 proves the following are equivalent:

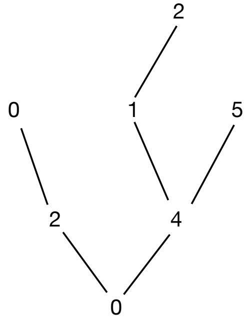
(1) Π^1_1-CA_0.

(2) $\widehat{\mathsf{WF}}$: If $\langle T_i \rangle_{i\in\mathbb{N}}$ is a sequence of trees with integer labeled nodes, then there is a function $f : \mathbb{N} \to 2$ such that $f(i) = 1$ if and only if T_i is well founded. (Lemma IV.1.1, Simpson [5])

(3) $\widehat{\mathsf{WF}}_L$: If $\langle T_i, L_i \rangle_{i\in\mathbb{N}}$ is a sequence of trees, each equipped with a leaf set L_i, then there is a function $f : \mathbb{N} \to 2$ such that $f(i) = 1$ if and only if T_i is well founded.

Leaf management

A tree can be converted to a tree with a leaf set by adding an extension with a new label to every existing nodes. The converted tree has the same infinite paths (and the same perfect subtrees).



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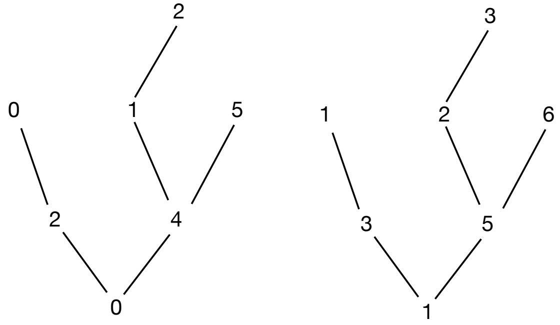

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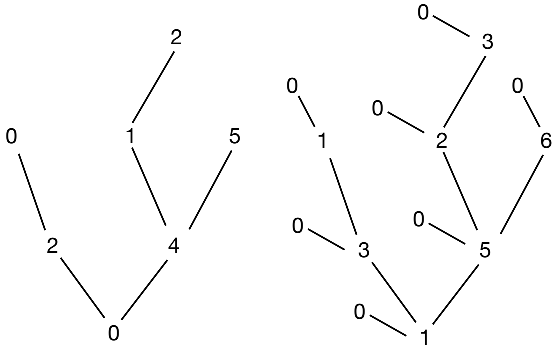
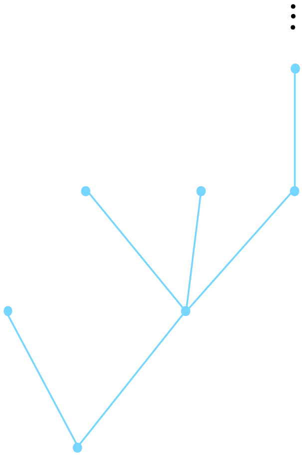


The reversal: $\widehat{\text{HC}} \to \widehat{\text{WF}}$

We want to convert a tree into a hypergraph that has a proper 2-coloring iff the tree has a path.



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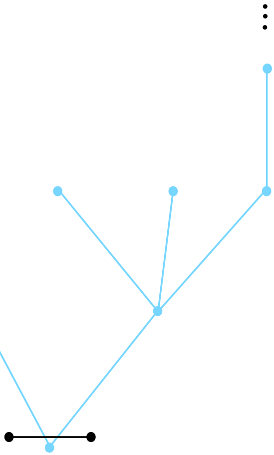

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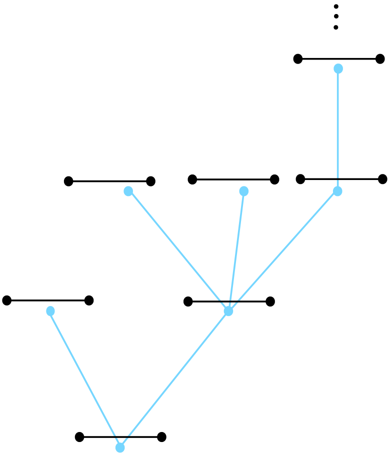

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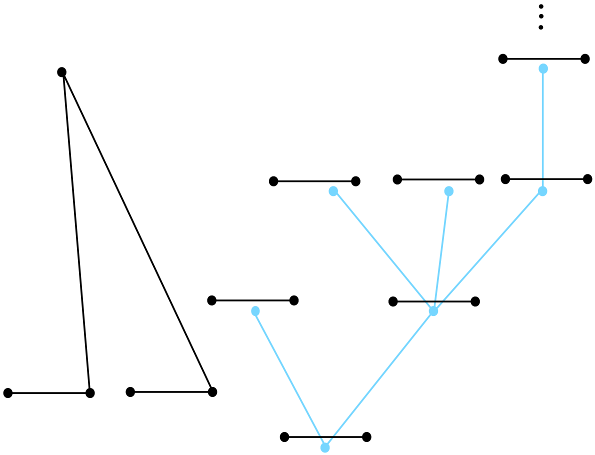

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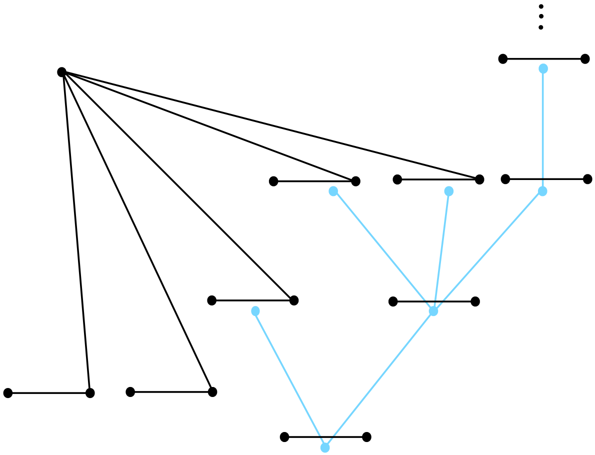

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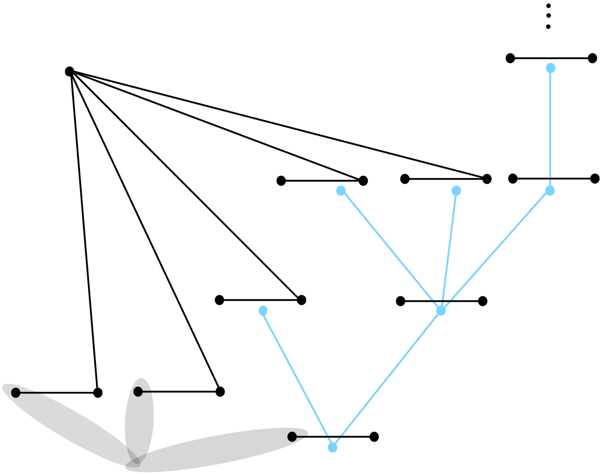

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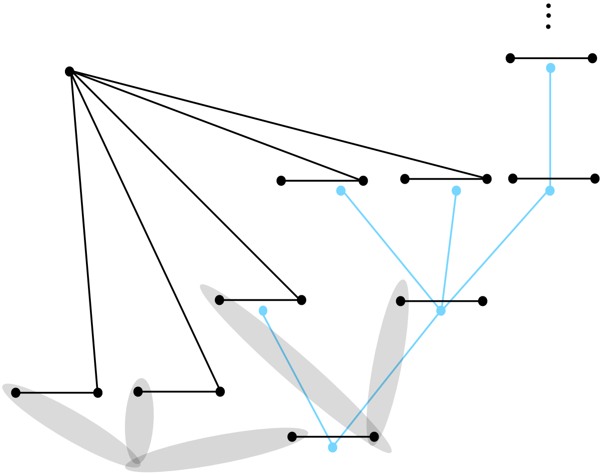

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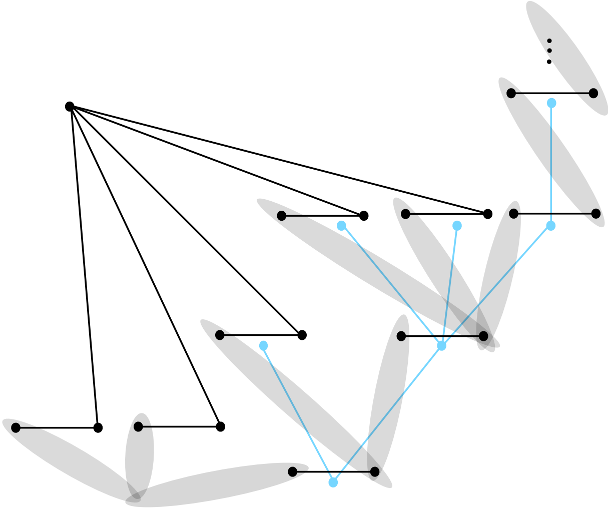

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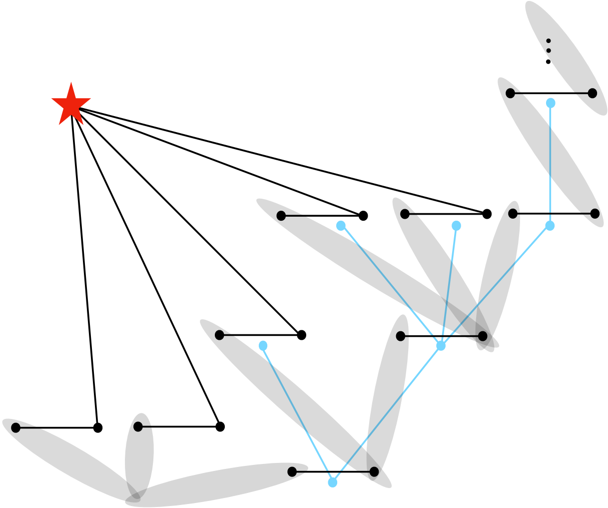

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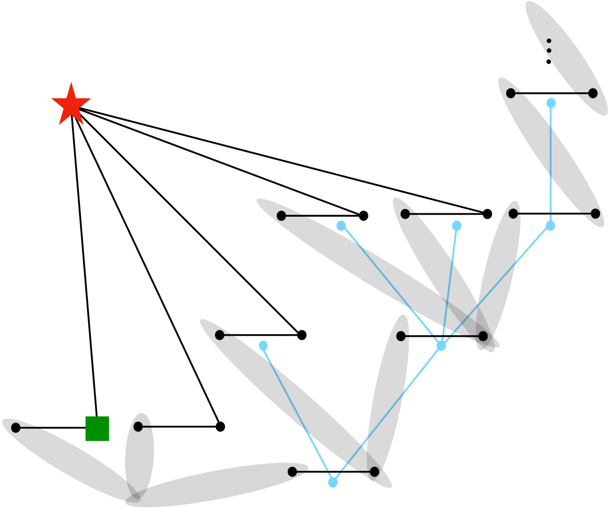

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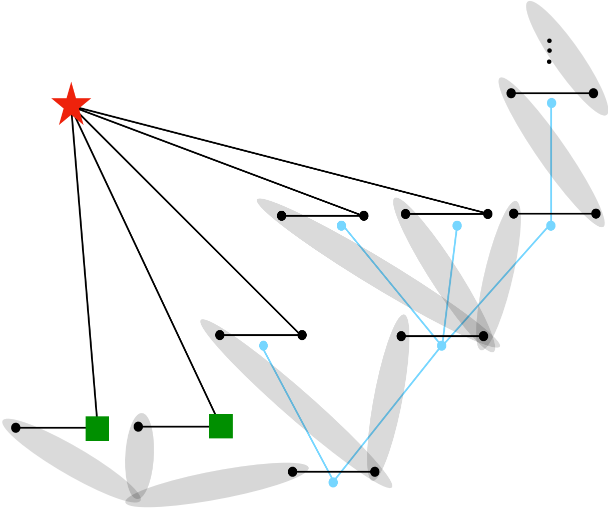

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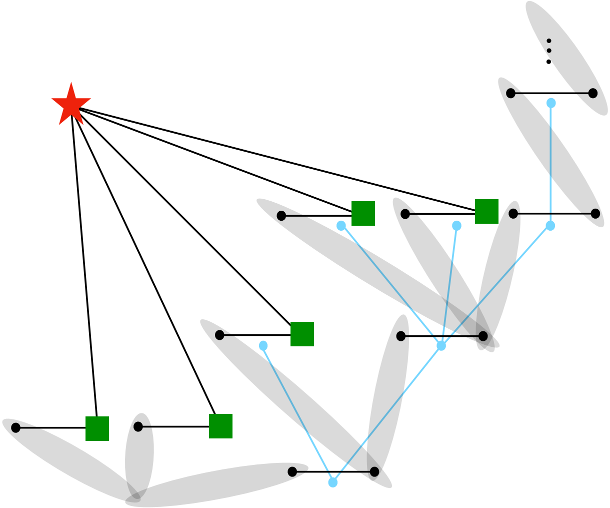

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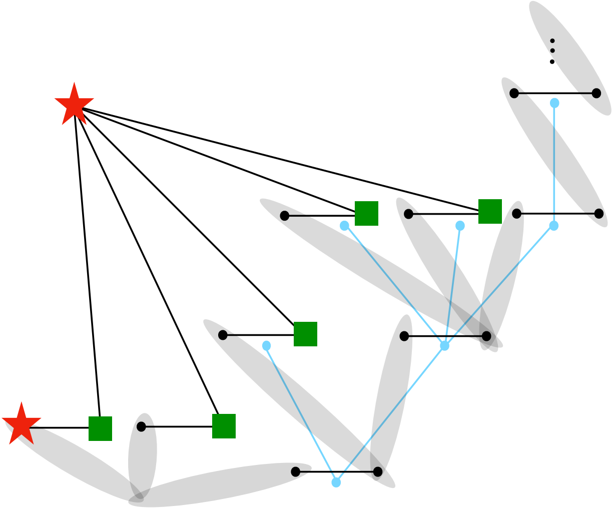

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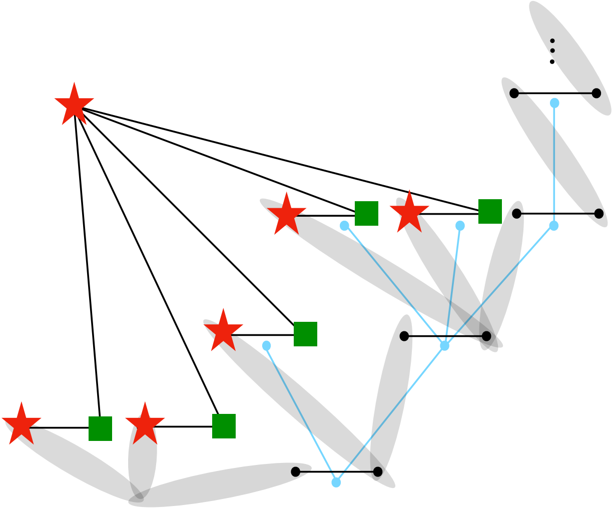

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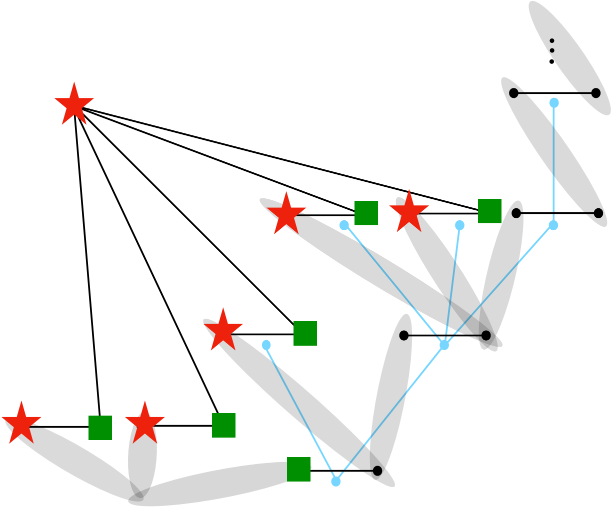

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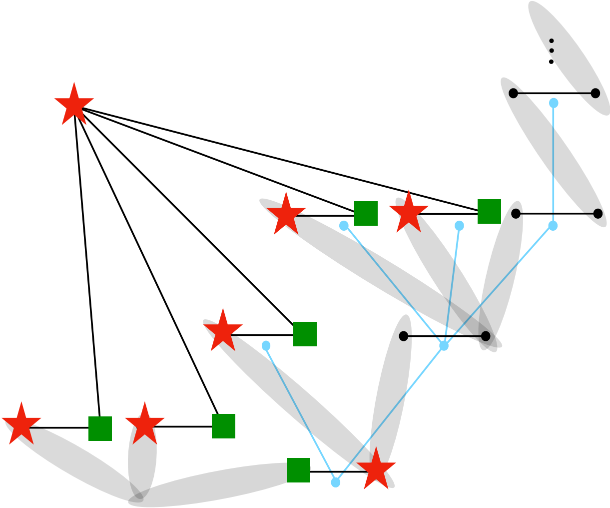
We want to convert a tree into a hypergraph that has a proper 2-coloring iff the tree has a path.



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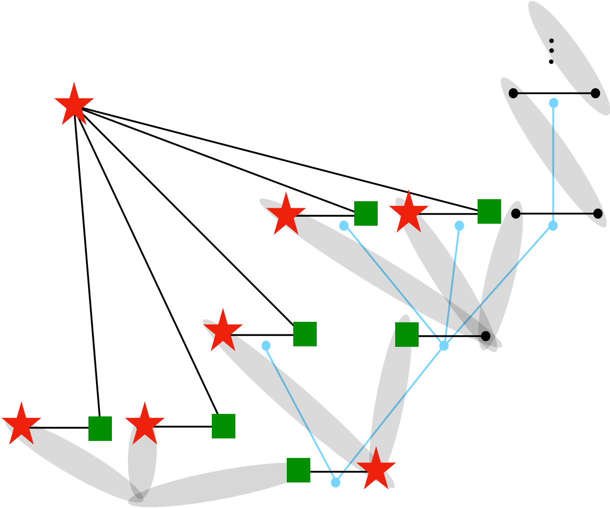

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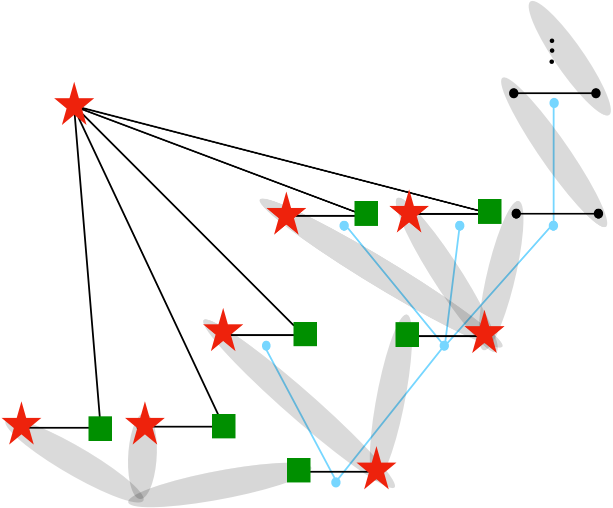

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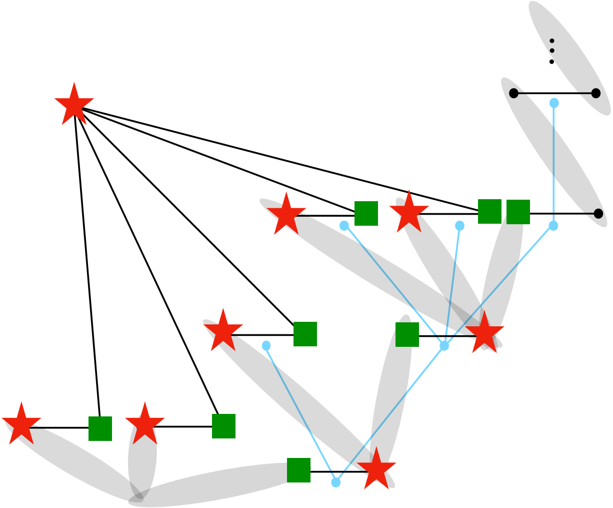

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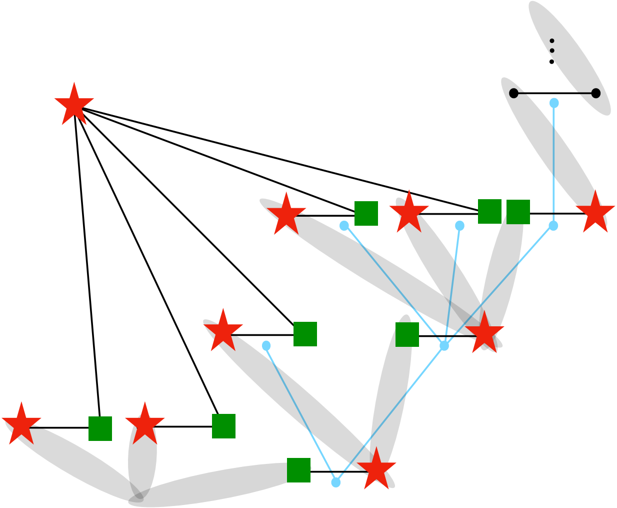

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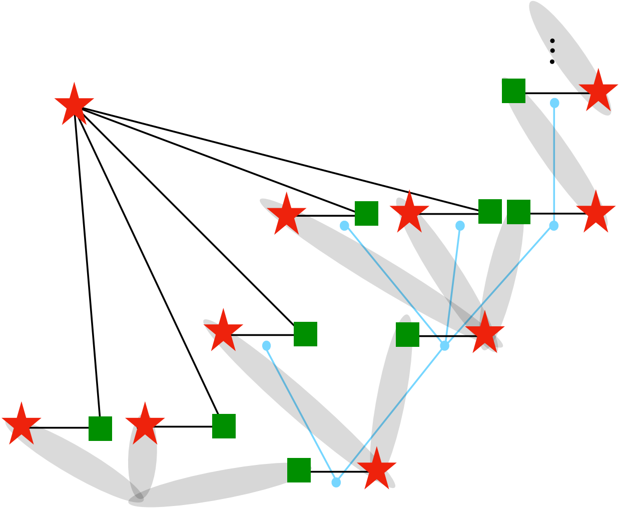

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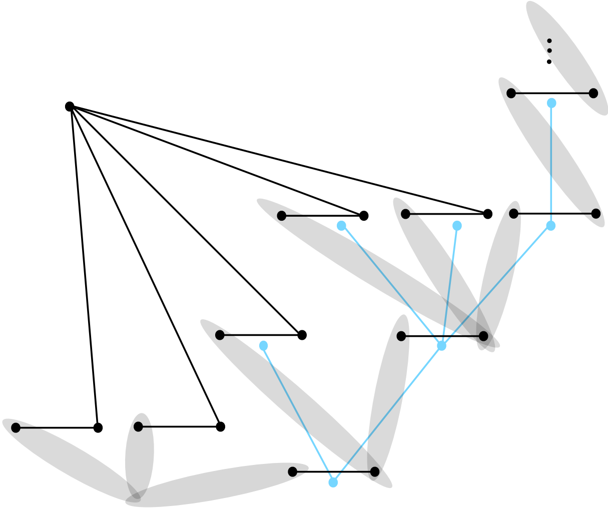

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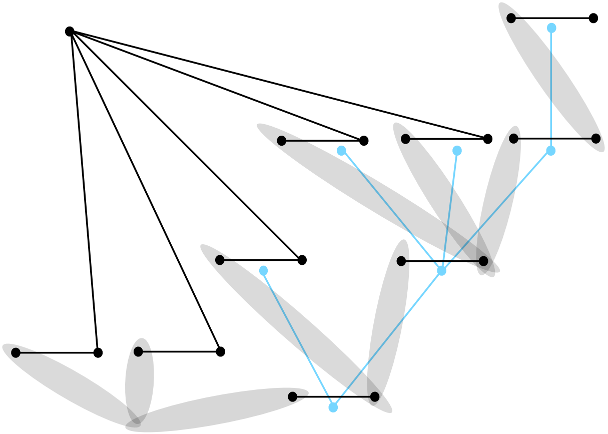

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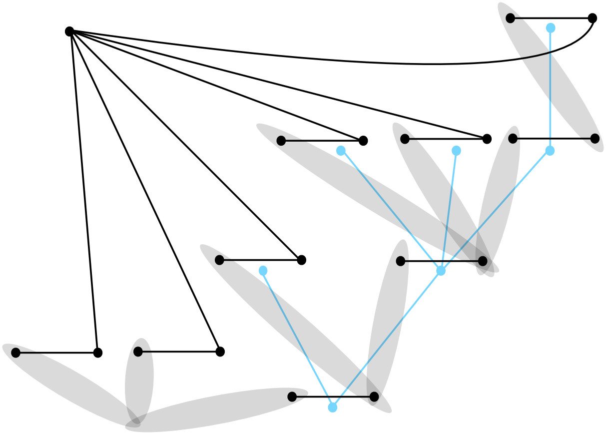

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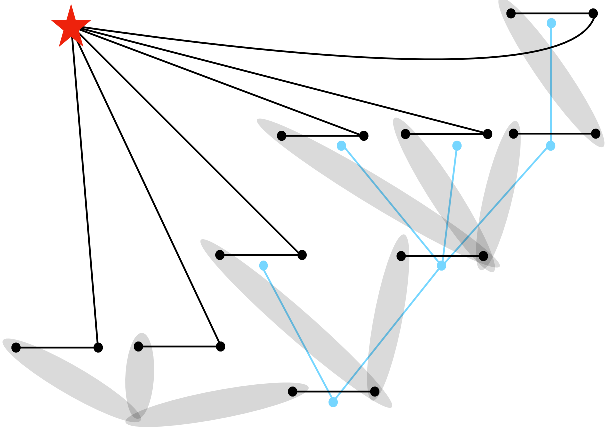

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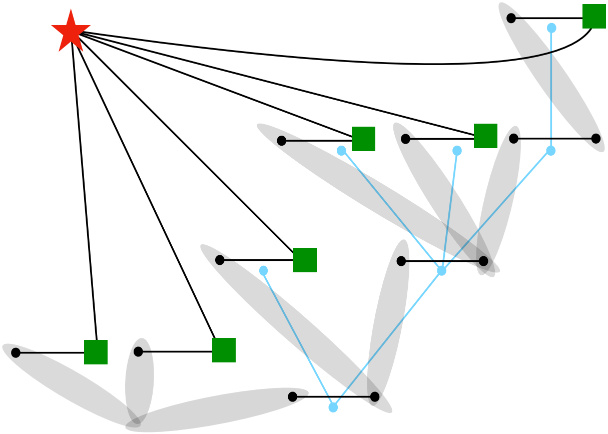

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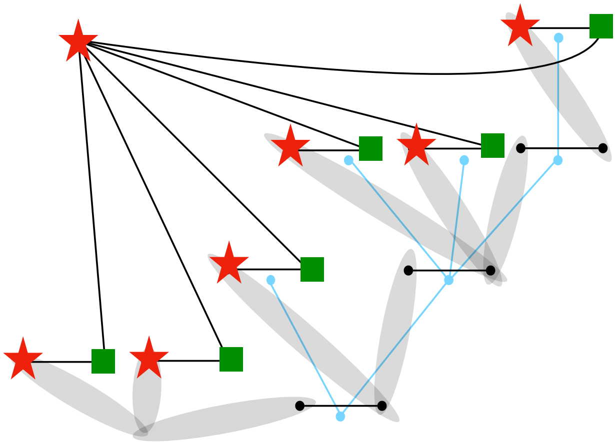

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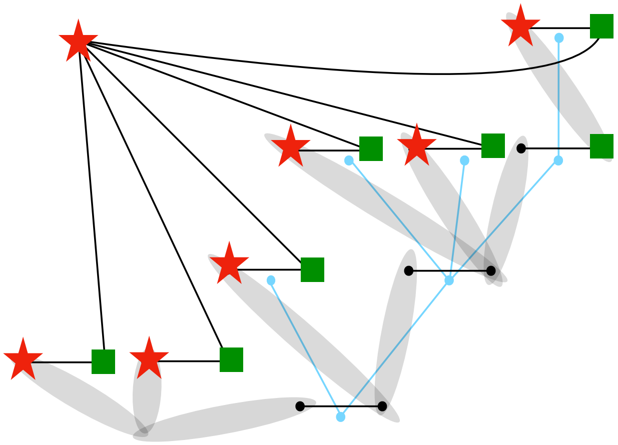

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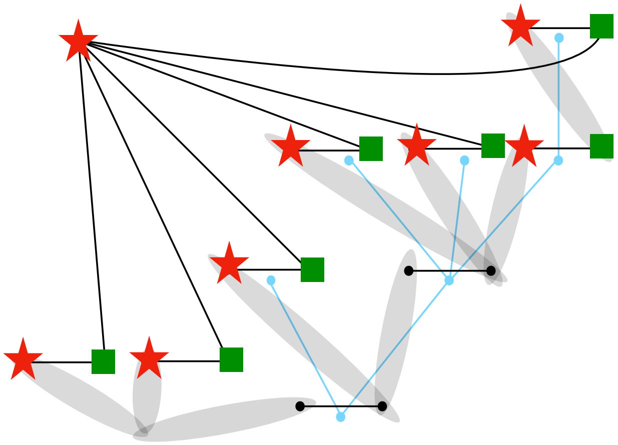

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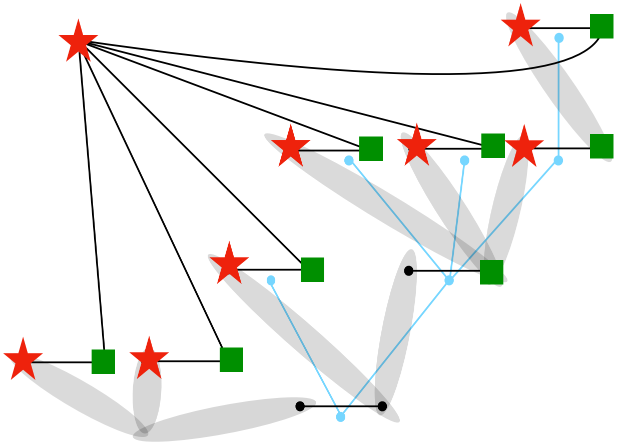

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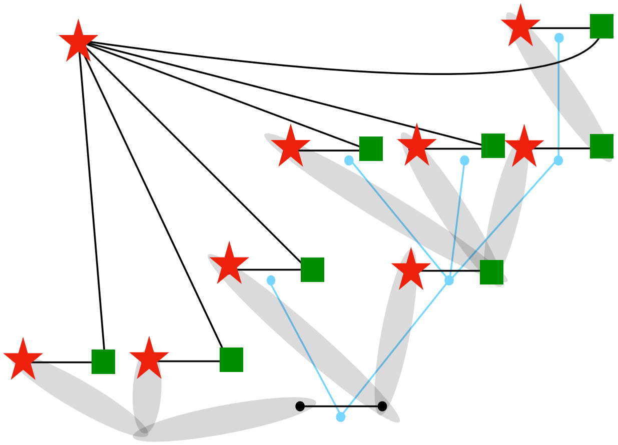

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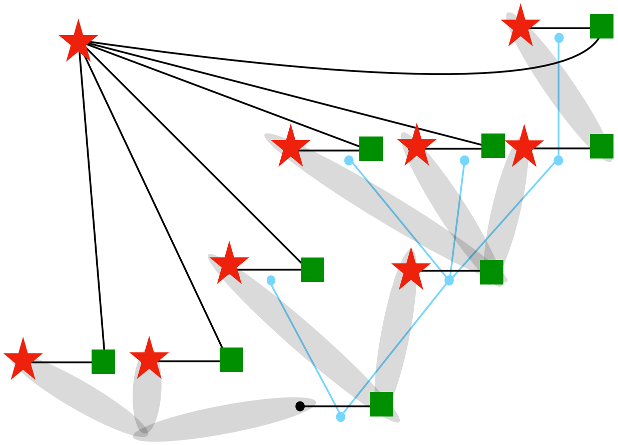

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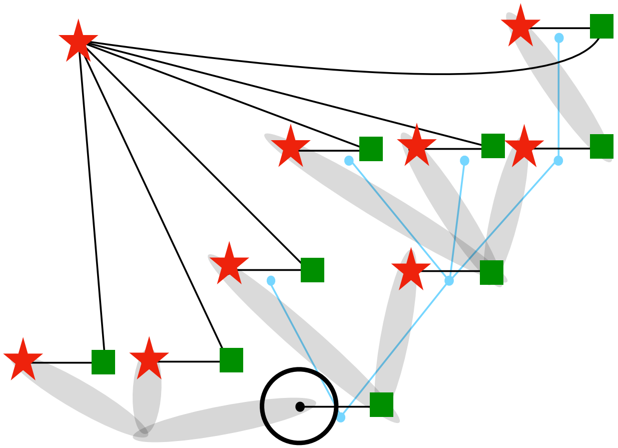

Weihrauch reductions

Sample problems

WF: input a tree T; output 1 iff T is well-founded.

HC: input a hypergraph H; output 1 iff H has a proper 2-coloring.

Parallelization

$\widehat{\mathrm{HC}}$: input an infinite sequence of hypergraphs; output list of indices of hypergraphs with proper 2-colorings.

Reductions

$\mathrm{P} \leqslant_{\mathrm{sW}} \mathrm{Q}$ if there are uniformly computable procedures φ and ψ such that

$$\begin{array}{ccc} \mathrm{P}_{\mathrm{input}} & \to_{\varphi} & \mathrm{Q}_{\mathrm{input}} \\ \downarrow & & \downarrow \\ \mathrm{P}_{\mathrm{output}} & \leftarrow_{\psi} & \mathrm{Q}_{\mathrm{output}} \end{array}$$

Equivalences

$\mathrm{P} \equiv_{\mathrm{sW}} \mathrm{Q}$ iff $\mathrm{P} \leqslant_{\mathrm{sW}} \mathrm{Q}$ and $\mathrm{Q} \leqslant_{\mathrm{sW}} \mathrm{P}$

Weihrauch equivalences

$$\mathsf{WF} \equiv_{sW} \mathsf{WF}_L \equiv_{sW} \mathsf{HC}$$

$$\widehat{\mathsf{WF}} \equiv_{sW} \widehat{\mathsf{WF}}_L \equiv_{sW} \widehat{\mathsf{HC}}$$

Another problem

PK: input a tree T; output the perfect kernel of T.

$$\widehat{\mathsf{WF}} \equiv_{sW} \mathsf{PK}$$

These results appear in *Leaf management* [4]

References

[1] Caleb Davis, Jeffry L. Hirst, Jake Pardo, and Tim Ransom, *Reverse mathematics and colorings of hypergraphs*, Archive for Mathematical Logic (2018).
DOI 10.1007/s00153-018-0654-z.

[2] Harvey Friedman, *Some systems of second order arithmetic and their use*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, 1975, pp. 235–242.
http://www.mathunion.org MR0429508.

[3] Harvey Friedman, *Abstracts: Systems of second order arithmetic with restricted induction, I and II*, J. Symbolic Logic **41** (1976), 557–559.
http://www.jstor.org/stable/2272259.

[4] Jeffry L. Hirst, *Leaf management*. To appear in Computability. Available at arxiv.org/abs/1812.09762.

[5] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, 2009.
10.1017/CBO9780511581007 MR2517689.