Reverse Mathematics: Constructivism and Combinatorics

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Principle Goals

- Contributions to the program of reverse mathematics
- Analysis of combinatorial theorems and proofs (especially Hindman's theorem)
- Explore relationships between reverse mathematics and constructive analysis

Relevance to the Call

- What are the limits of mathematics in advancing human knowledge?
- Contribute to our understanding of the limits of mathematics within mathematics.

Three forms of Hindman's Theorem

1. Suppose we have a finite coloring of the natural numbers. There is an infinite set $X \subset \mathbb{N}$ and a color j such that sum of each finite subset of X is a number colored j.

2. Suppose we have a finite coloring of the finite subsets of $\mathbb{N}$. There is an infinite collection of disjoint finite subsets X and a color j such that the union of each finite subset of X is colored j.

3. There is an almost downward translation invariant ultrafilter on the power set of $\mathbb{N}$.

   That is, there is an ultrafilter U such that for every set $X \in U$ there is an $n \in X$ such that difference set $X - n = \{x - n \mid x \in X\}$ is also in U.

History of Proofs of Hindman's Theorem

1972 Hindman proves the equivalence of the sum version and the ultrafilter version in in $\mathsf{ZFC} + \mathsf{CH}$ [3].

The sum version was conjectured by Graham and Rothschild. The ultrafilter version was formulated by Galvin.

1974 Hindman proves the sum version directly [4].

1975 Galvin and Glazer prove the ultrafilter version directly. The proof first appears in [2], the date is from [5].

History of reverse mathematics of Hindman's Theorem

1984 Blass, Hirst, and Simpson [1] analyze Hindman's direct combinatorial proof (union form)

- ACA_0^+ suffices to prove Hindman's theorem (and an iterated form of Hindman's theorem)
- Over RCA_0, Hindman's theorem implies ACA_0

2004 Hirst [6] analyzes the ultrafilter equivalence result

- RCA_0 proves that the iterated form of Hindman's theorem is equivalent to the existence of almost downward translation invariant ultrafilters on countable Boolean algebras closed under shift

Recent Developments

2011 Henry Towsner [8] publishes a formalization of an ultrafilter proof of Hindman's theorem in $\Pi^1_1 - \mathsf{TR}_0$

201? Towsner [9] publishes a simplified combinatorial proof of Hindman's theorem, formalized in ACA^+

Best current bounds

Hindman's theorem proves ACA_0 and is provable in ACA_0^+

An ultrafilter based proof can be carried out in $\Pi^1_1 - \mathsf{TR}_0$

Plans

Modifying Towsner's new combinatorial proof so that it can be formalized in ACA_0.

Using ideas from the Towsner proof modifications, revisit the analysis of Hindman's combinatorial proof, trying to formalize it in ACA_0.

Examine methods of expanding countable shift algebras in such a way that Glazer's ultrafilter addition becomes well behaved.

Use the preceding to carry out the Galvin/Glazer proof in ACA_0 or ACA_0^+.

Collaborators: Dzhafarov (Notre Dame), Hirst (Appalachian State University), Mummert (Marshall University), and Towsner (University of Connecticut)

Where's the constructivism?

Hirst and Mummert [7] proved:

Theorem. *Let* $\forall x \exists y A(x,y)$ *be a sentence of* $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ *in* Γ_1. *If*

$$\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}} \vdash \forall x\, \exists y\, A(x,y),$$

then

$$\mathsf{RCA}^\omega \vdash \forall \langle x_n \mid n \in \mathbb{N}\rangle\, \exists \langle y_n \mid n \in \mathbb{N}\rangle\, \forall n\, A(x_n, y_n).$$

Furthermore, if x *and* y *are both type* 1 *(set) variables, and the formula* $\forall x\, \exists y A(x,y)$ *is in* $\mathcal{L}(\mathsf{RCA}_0)$, *then* RCA^ω *may be replaced by* RCA *in the implication.*

Questions

1. Can computable restrictions of strong uniformizations help formulate results in constructive analysis?
2. What additional information can the strength of uniformizations provide about nonconstructive theorems?

Collaborators: Dorais (ASU), Hirst (ASU), Mummert (Marshall), Shafer (ASU)

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