More Reverse Mathematics of the Heine-Borel Theorem

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History

One of Friedman's earliest results in reverse mathematics:

Theorem

(RCA_0) *The following are equivalent:*

1. $\mathsf{HB}([0,1])$ *The Heine-Borel theorem for* $[0,1]$*: Every open cover of a closed subset of* $[0,1]$ *contains a finite subcover.*
2. WKL_0 *Weak König's Lemma: Every infinite* $0-1$ *tree contains an infinite path.*

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What other countable subsets of $[0,1]$ work?

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We want two notions $\mathsf{W}(X)$ and $\mathsf{S}(X)$ satisfying:

- $\mathsf{RCA}_0 \vdash \mathsf{S}(X) \to (\mathsf{HB}(X) \to \mathsf{WKL}_0)$
- $\mathsf{RCA}_0 \vdash \mathsf{W}(X) \to \mathsf{HB}(X)$
- Some sufficiently strong system will prove $\forall X(\mathsf{S}(X) \vee \mathsf{W}(X))$.

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The definition
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Ideas from the proof

- If S(X) then X contains a set analogous to the midpoints of the Cantor middle third intervals.

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- If $\mathsf{S}(X)$ then X contains a set analogous to the midpoints of the Cantor middle third intervals.
- Given a tree with no infinite paths, the associated "midpoints" form a closed set, with a natural cover.
- Any finite cover witnesses that the tree is finite.

W(X) Heine-Borel theorem for X is weak.

The definition
W(X)*denotes* : X is contained in a countable closed subset $F \subseteq [0,1]$ and there are functions f and g and a well ordering Y satisfying:

- The function $f : F \to Y$ is one to one.
- For any $b_1, b_2 \in (\mathbb{Q} \cap [0,1]) \cup \{-.1, 1.1\}$ with $b_1 < b_2$, if $F - (b_1, b_2)$ is nonempty, then the value of $g(b_1, b_2)$ is an element of $F - (b_1, b_2)$ such that

$$\forall x \in F - (b_1, b_2)\ f(x) \leqslant f(g(b_1, b_2)).$$

Roughly W(X) says X is contained in a closed subset that can be well ordered in such a way that its nicely defined closed subsets have easily calculated maximums.

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(RCA$_0$) *For all X, if* W(X) *then* HB(X).

Ideas from the proof

- Suppose W(X) holds. Let A be a closed subset of X and pick a (countable) cover of A. Add (nice open subsets of) the complement of A to get a cover of F.

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- Pick the (well-ordering) maximum element of F. Pick the first open set in the cover that contains it.

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- Pick the maximum in the remainder of F. (It's smaller than the previous one.) Pick the first open set in the cover that contains it.

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- Pick the (well-ordering) maximum element of F. Pick the first open set in the cover that contains it.
- Pick the maximum in the remainder of F. (It's smaller than the previous one.) Pick the first open set in the cover that contains it.
- Iterate. Since F is well-ordered, the process halts, yielding the desired finite subcover.

The remaining goal

Theorem

(ACA_0) *The following are equivalent:*

1. ATR_0.
2. *For every closed subset of* $[0,1]$, *exactly one of* $\mathsf{W}(X)$ *and* $\mathsf{S}(X)$ *holds.*

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Question: Are there other characterizations for $W(X)$ and $S(X)$ that would

- satisfy the goals,
- guarantee that the classes $\{X \mid S(X)\}$ and $\{X \mid W(X)\}$ are subsets in submodels, and
- all us to prove the last theorem in weaker subsystems?

Bibliography

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