Counting uses of a theorem

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Motivating question:

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$\mathsf{RT}(2,n)$ is Ramsey's theorem for pairs and n colors:

Given $f : ([\mathbb{N}]^2) \to n$, we can find an infinite $X \subset \mathbb{N}$ and an $i < n$ such that $f([X]^2) = i$.

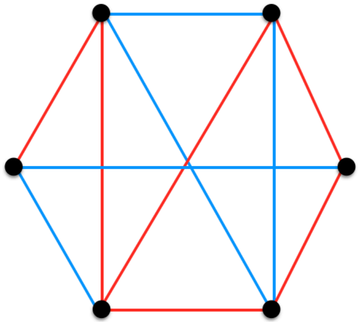
Vocabulary:

f is a coloring with n colors.

X is an infinite monochromatic set.

Pictures related to RT$(2,2)$

Theorem: (RT$(2,2)$) If G is the complete graph with vertices $V = \{v_0, v_1, \ldots\}$, and $f : [V]^2 \to \{\text{red, blue}\}$ colors the edges of G, then there is an infinite $S \subset V$ such that the subgraph with vertices from S is monochromatic.



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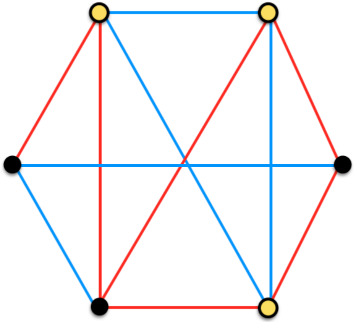

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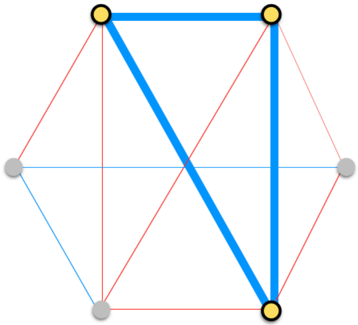

Further motivation

RCA_0 can prove $\mathsf{RT}(2,4)$ with two uses of $\mathsf{RT}(2,2)$.

Given $f : [\mathbb{N}]^2 \to 4$, define:

$$g_1(m,n) = \begin{cases} 0 & \text{if } f(m,n) \in \{0,1\} \\ 1 & \text{if } f(m,n) \in \{2,3\} \end{cases}$$

Let $X = \{x_0, x_1, \ldots\}$ be an infinite monochromatic set for g_1.
Note that $f([X]^2) \in \{0,1\}$ or $f([X]^2) \in \{2,3\}$.

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Note that $f([X]^2) \in \{0,1\}$ or $f([X]^2) \in \{2,3\}$. Define:

$$g_2(m,n) = \begin{cases} 0 & \text{if } f(x_m,x_n) \text{ is even} \\ 1 & \text{if } f(x_m,x_n) \text{ is odd} \end{cases}$$

Let Y be an infinite monochromatic set for g_2. Then
$Z = \{x_m \mid m \in Y\}$ is monochromatic for f.

Motivating question:

Can you prove $\mathsf{RT}(2,4)$ with one use of $\mathsf{RT}(2,2)$?

Answers:

Motivating question:

Can you prove $\mathrm{RT}(2,4)$ with one use of $\mathrm{RT}(2,2)$?

Answers:

NO

YES

Motivating question:

Can you prove $\mathrm{RT}(2,4)$ with one use of $\mathrm{RT}(2,2)$?

Answers:

NO: The intuitionistic system $i\mathrm{RCA}_0^\omega$ cannot prove $\mathrm{RT}(2,4)$ with one typical use of $\mathrm{RT}(2,2)$.

YES: RCA_0 can prove $\mathrm{RT}(2,4)$ with one use of $\mathrm{RT}(2,2)$.

One is not enough: Vocabulary

An axiom system formulated by Kohlenbach [3]

$i\mathrm{RCA}_0^\omega$ includes:

$\widehat{\text{E-HA}}^\omega_\restriction$	Intuitionistic arithmetic in all finite types with restricted induction and primitive recursion
$\text{QF-AC}^{1,0}$	A choice scheme that implies the recursive comprehension axiom (RCA)

Note: Intuitionistic systems cannot prove
all instances of the law of the excluded middle: $\neg A \vee A$

We consider *problems* of the form $\mathrm{P} : \forall x (p_1(x) \to \exists y\, p_2(x, y))$

$p_1(x)$	means x is an instance of the problem P
$p_2(x, y)$	means y is a solution of the instance x of the problem P.

One is not enough: What is does recursive mean?

A function $f : \mathbb{N} \to \mathbb{N}$ is recursive (or computable) if there is a deterministic algorithm that for every input number eventually halts and outputs a number.

A set $x \subset \mathbb{N}$ is recursive if there is a recursive function f such that $x = \{n \in \mathbb{N} \mid f(n) \neq 0\}$.

The **R**ecursive **C**omprehension **A**xiom formalizes: if f is a recursive function, then the set $x = \{n \in \mathbb{N} \mid f(n) \neq 0\}$ exists. (Note: Actually, RCA asserts the existence of sets that are recursive relative to known sets.)

The recursive comprehension axiom is the main set existence axiom in RCA_0, the base axiom system for reverse mathematics.

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One is not enough: Weihrauch reductions

We consider *problems* of the form $\mathrm{P} : \forall x (p_1(x) \rightarrow \exists y\, p_2(x, y))$

$p_1(x)$	means x is an instance of the problem P
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In this setting $\mathrm{Q} \leqslant_W \mathrm{P}$ means there are computable functionals ψ and ϕ such that

$$\begin{array}{ccc} & \psi & \\ x_Q & \longrightarrow & x_P \\ \downarrow & & \downarrow \\ y_Q & \longleftarrow & y_P \\ & \phi & \end{array}$$

Note: ϕ can use information about x_Q to compute y_Q. (This is weak reduction.)

One is not enough: Formalized Weihrauch reductions

Given problems:
$\mathsf{P} : \forall x(p_1(x) \to \exists y\, p_2(x,y))$ and $\mathsf{Q} : \forall x(q_1(x) \to \exists y\, q_2(x,y))$

in the language of $i\mathsf{RCA}_0^\omega$, we use $\mathsf{Q} \leqslant_W \mathsf{P}$ to abbreviate

$$\exists \varphi \exists \psi \forall u\, (q_1(u) \to (p_1(\varphi(u)) \wedge \forall y[p_2(\varphi(u), y) \to q_2(u, \psi(u,y))]))$$

Which says that there are functionals φ and ψ such that

$q_1(u)$	If u is an instance of Q
$p_1(\varphi(u))$	then $\varphi(u)$ is an instance of P
$p_2(\varphi(u), y)$	such that whenever y is a solution the instance $\varphi(u)$ of the problem P
$q_2(u, \psi(u,y))$	$\psi(u,y)$ computes a solution to the instance u of the problem Q

In computability theory, φ and ψ are computable functionals

One is not enough: What does one mean?

A theory proves Q with one typical use of P if

From $q_1(u)$ we can deduce the existence of x_u, an instance of P.

From $p_1(x_u) \to \exists y p_2(x_u, y)$ we can deduce the existence of $v_{x_u,y}$ with $q_2(u, v_{x_u,y})$.

Theorem: If P and Q are nice problems then $i\mathrm{RCA}_0^\omega$ proves Q with one typical use of P if and only if $i\mathrm{RCA}_0^\omega \vdash \mathrm{Q} \leqslant_W \mathrm{P}$.

One is not enough: What does one mean?

A theory proves Q with one typical use of P if

- From $q_1(u)$ we can deduce the existence of x_u, an instance of P.
- From $p_1(x_u) \rightarrow \exists y p_2(x_u, y)$ we can deduce the existence of $v_{x_u,y}$ with $q_2(u, v_{x_u,y})$.

Theorem: If P and Q are nice problems then $i\mathrm{RCA}_0^\omega$ proves Q with one typical use of P if and only if $i\mathrm{RCA}_0^\omega \vdash \mathrm{Q} \leqslant_W \mathrm{P}$.

Nice problems: $q_1(u) \rightarrow (p_1(x) \wedge [p_2(x, y) \rightarrow q_2(u, v)])$ is in Γ_1

Proof uses modified reducibility [2], adapted from Kohlenbach.

Rutger Kuyper [4] has proved related results.

One is not enough: The main result

Corollary:
$i\mathsf{RCA}_0^\omega$ cannot prove $\mathsf{RT}(2,4)$ with one typical use of $\mathsf{RT}(2,2)$.

Proof in three steps:

1: $\mathsf{RT}(2,4) \not\leqslant_W \mathsf{RT}(2,2)$ Dorais, Shafer et al [1]

2: $i\mathsf{RCA}_0^\omega \nvdash \mathsf{RT}(2,4) \leqslant_W \mathsf{RT}(2,2)$
$i\mathsf{RCA}_0^\omega$ lies only by omission

3: Apply the previous theorem.

Sometimes, one is enough

Claim: RCA_0 can prove $\mathsf{RT}(2,4)$ with one use of $\mathsf{RT}(2,2)$.

Sketch: Given $f : [\mathbb{N}]^2 \to 4$, either

- $\exists X$ infinite with $f([X]^2) \subset \{a_0, a_1\} \subset \{0, 1, 2, 3\}$

  or
- there is no such X.

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Sketch: Given $f : [\mathbb{N}]^2 \to 4$, either

- $\exists X$ infinite with $f([X]^2) \subset \{a_0, a_1\} \subset \{0, 1, 2, 3\}$

  Fix such an $X = \{x_0, x_1, \dots\}$ and set $j = 0$
- there is no such X.

  Let $X = \mathbb{N}$ and set $j = 1$

Sometimes, one is enough

Claim: RCA_0 can prove $\mathsf{RT}(2,4)$ with one use of $\mathsf{RT}(2,2)$.

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  Fix such an $X = \{x_0, x_1, \dots\}$ and set $j = 0$
- there is no such X.

  Let $X = \mathbb{N}$ and set $j = 1$

Define $g : [\mathbb{N}]^2 \to 2$ by:

$$g(m,n) = \begin{cases} 0 & \text{if } j = 0 \wedge f(x_m, x_n) = a_0 \\ 1 & \text{if } j = 0 \wedge f(x_m, x_n) = a_1 \\ 0 & \text{if } j = 1 \wedge f(x_m, x_n) \in \{0,1\} \\ 1 & \text{if } j = 1 \wedge f(x_m, x_n) \in \{2,3\} \end{cases}$$

If Y is mono. for g, then $j = 0$ and $\{x_m \mid m \in Y\}$ is mono. for f.

Some references

[1] François G. Dorais, Damir D. Dzhafarov, Jeffry L. Hirst, Joseph R. Mileti, and Paul Shafer, *On uniform relationships between combinatorial problems*, Trans. AMS **368** (2014), 1321–1359. DOI 10.1090/tran/6465.

[2] Jeffry L. Hirst and Carl Mummert, *Reverse mathematics and uniformity in proofs without excluded middle*, Notre Dame J. Form. Log. **52** (2011), no. 2, 149–162. DOI 10.1215/00294527-1306163.

[3] Ulrich Kohlenbach, *Higher order reverse mathematics*, Reverse Mathematics 2001 (Stephen G. Simpson, ed.), Lect. Notes Log., vol. 21, Assoc. Symbol. Logic, La Jolla, CA, 2005, pp. 281–295.

[4] Rutger Kuyper, *On Weihrauch reducibility and intuitionistic reverse mathematics*. Preprint, arXiv:1511.05189v1.