Reverse Mathematics and Brouwer's Fixed Point Theorem

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Introduction

In 1949, B.H. Arnold (Oregon State U.) published a proof of the fundamental theorem of algebra using Brouwer's fixed point theorem.

Fundamental theorem of algebra:
Every nonconstant polynomial with complex coefficients has a zero.

Brouwer's fixed point theorem:
If $f : I^2 \to I^2$ is continuous, then for some $z \in I^2$, $f(z) = z$. In general, any continuous map of a compact, convex space to itself has a fixed point.

Reverse Mathematics

Theorem: (Simpson [4]) $\mathbf{RCA_0}$ can prove the fundamental theorem of algebra.

Theorem: (Shioji and Tanaka [3]) $\mathbf{RCA_0}$ proves that these are equivalent:

1. Brouwer's fixed point theorem for I^2.
2. $\mathbf{WKL_0}$.

Conclusions:

1. Brouwer's theorem is a very big hammer to use on the FTA.
2. There should be a restricted (computable) version of Brouwer's thm that could be used in Arnold's proof.

Terminology for a computable fixed point theorem

A *computably coded continuous function*: is encoded by a computable set of 5-tuples, each of which defines a δ neighborhood and an ϵ neighborhood that contains $f(\delta)$.

A *modulus of uniform continuity for* f: is a function h such that for every n,

$$|x - y| < 2^{-h(n)} \to |f(x) - f(y)| < 2^{-n}.$$

We write $f : I^2 \to I^2$ if a ccc function f is defined at every computable point in I^2.

Given $f : I^2 \to I^2$ we define its *extension* $f^\star$ by setting $f^\star(a) = \lim_{x \to a} f(x)$ at each a where the limit exists, and saying $f^\star$ is undefined at other points.

A computable restriction of Brouwer's fixed point theorem:

Suppose that

(1) $f : I^2 \to I^2$ is a computably coded continuous function,

(2) f has a computable modulus of uniform continuity, and

(3) $f^\star$ has finitely many fixed points.

Then f has a computable fixed point.

The proof is based on:

Lemma: Suppose that

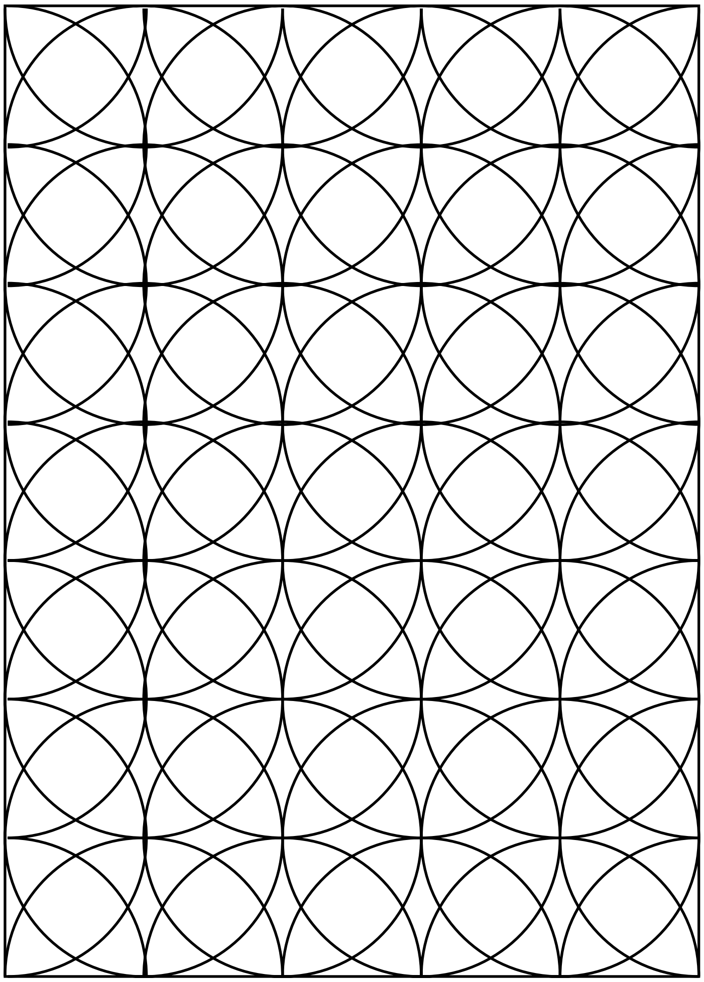
(1) $f : I^2 \to I^2$ is a computably coded continuous function, and

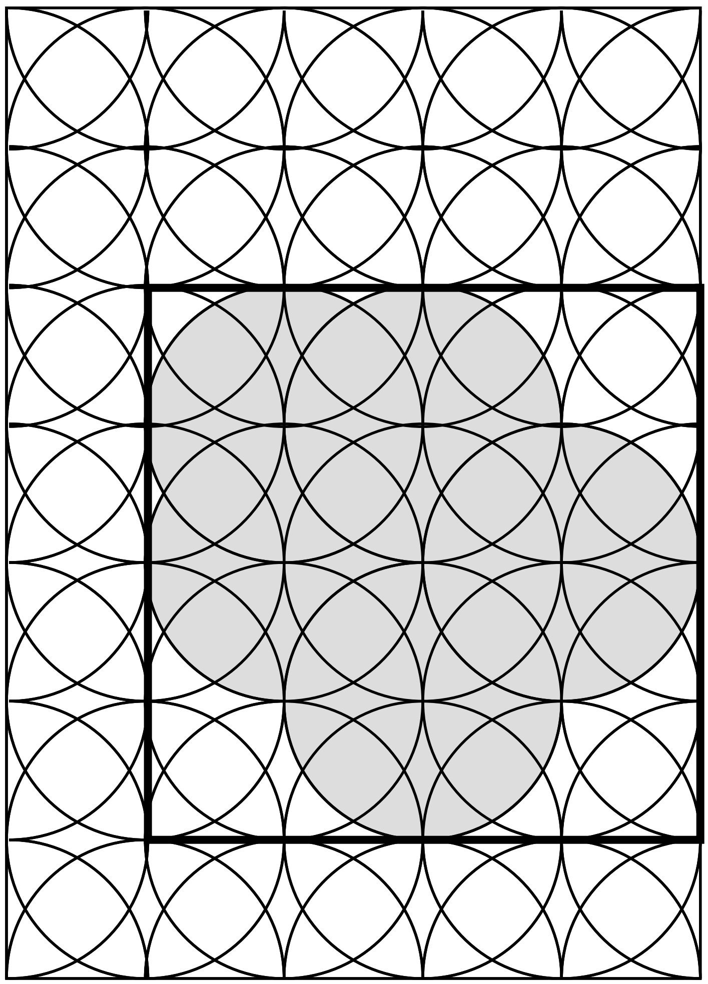
(2) f has a computable modulus of uniform continuity.

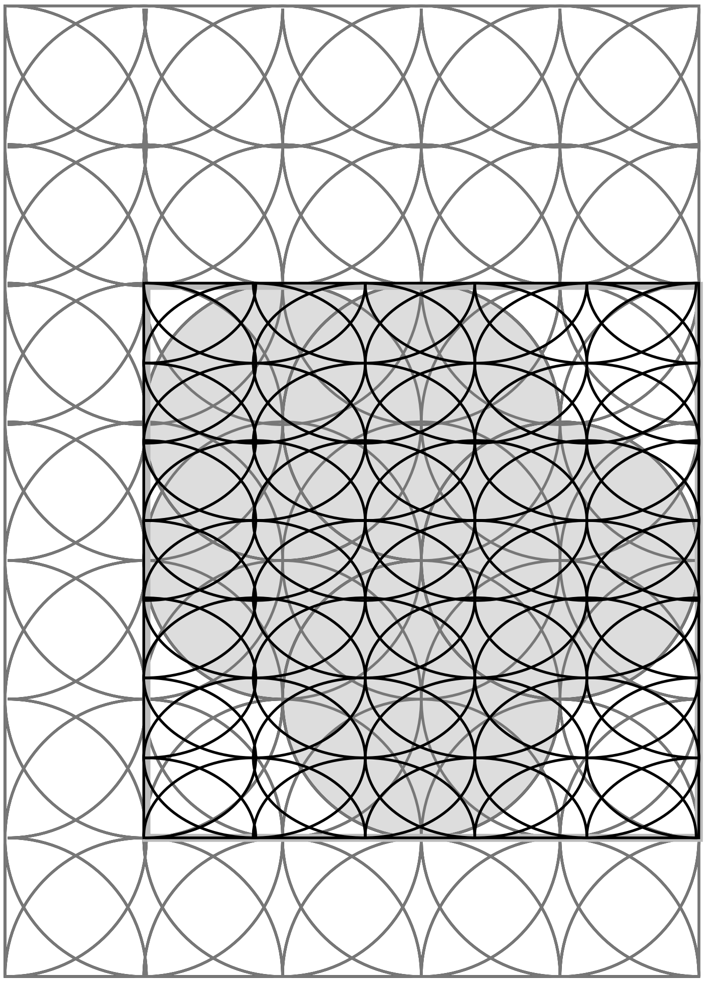
Then every isolated fixed point of $f^\star$ is a computable fixed point of f.

To compute
an isolated zero of $g(z) = |f(z) - z|$,

- isolate the zero in a rectangle,
- cover the rectangle with δ nhoods,
- mark the nhoods where g(center point) is very close to 0,
- draw a new smaller rectangle containing the marked nhoods.







Orevkov's Construction

In [2], V.P. Orevkov constructs a ccc function $f : I^2 \to I^2$ such that $f^\star$ has no fixed points.

In this construction:

$f^\star$ is not total, and

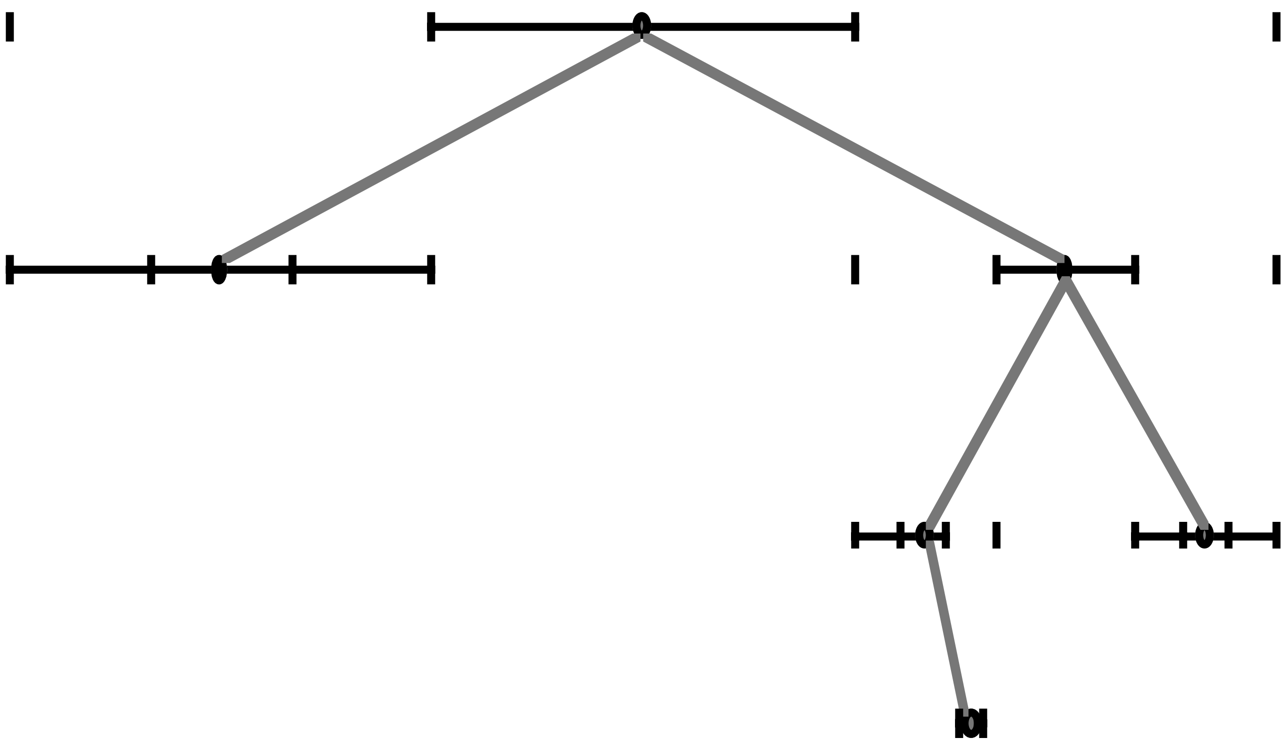
f has no computable modulus of uniform continuity.

Building computable counterexamples

Let T be an infinite computable 0-1 tree with no computable paths.

Using T, we can build a computable sequence of closed subintervals of $[0, 1]$ such that

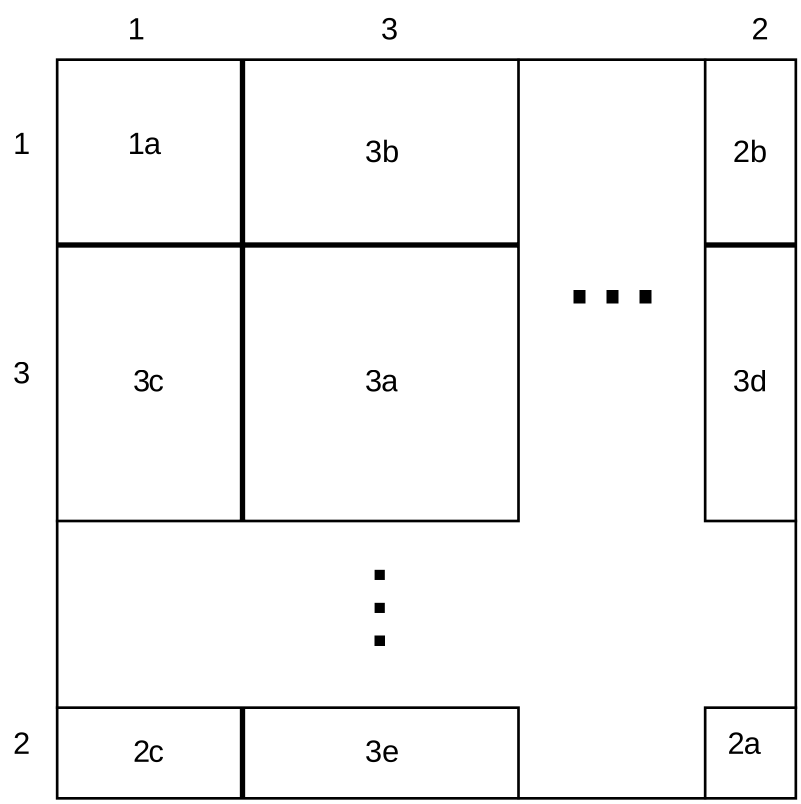
- each pair of intervals intersect in at most one point,
- each computable real in $[0, 1]$ is contained in one of the intervals, and
- there is a degree preserving isomorphism between the points of $[0, 1]$ not contained in the union of the intervals and the paths through T.



The picture for Orevkov's construction

To build a function with no fixed points:

- divide the square using subintervals generated by a tree,
- map each square to the boundary of I^2,
- rotate.

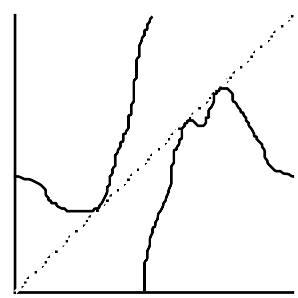


Question: If $f : I^2 \to I^2$ is ccc and $f^\star$ is total must f have a computable fixed point?

Partial answer:

Theorem: If $f : I^2 \to \partial I^2$ is ccc and $f^\star$ is total, then f has a computable fixed point.

Sketch: If $f : D \to \partial D$ and $f^\star$ is total, then $f^\star$ has a fixed point on ∂D. Graph $f^\star : \partial D \to \partial D$ as a function of radians.

This sort of thing can't happen: 

So $f^\star$ must cross $y = x$ someplace. Apply the computable version of the IVT to find the fixed point.

Theorem: Given any infinite computable tree T with no computable paths, there is a computably coded continuous function $f : I^2 \to \partial I^2$ such that $f^\star$ is total, the only computable fixed point of $f^\star$ is $(0, 0)$, and there is a degree preserving isomorphism between the noncomputable fixed points of $f^\star$ and the infinite paths through T.

Sketch: Fix T. Construct a computably coded continuous function $g : [0, 1] \to [0, 1]$ such that the maximum of g is 1, and there is a degree preserving isomorphism between $\{x \mid g^\star(x) = 1\}$ and the infinite paths through T.

Define $f(x, y) = (x \cdot g(x), 0)$. The only fixed points of f occur where $y = 0$ and either $x = 0$ or $g(x) = 1$.

The original task...

We wanted a restricted version of Brouwer's Theorem that could be used in $\mathbf{RCA_0}$ to formalize Arnold's proof of the fundamental theorem of algebra.

Our computable version uses the condition "$f^\star$ has finitely many fixed points."

We could use this to produce a computable analog of Arnold's proof, but $f^\star$ can't be formalized in $\mathbf{RCA_0}$.

We need another version of the fixed point theorem.

Theorem ($\mathbf{RCA_0}$) Suppose that:

(1) $f : I^2 \to I^2$ is a total continuous function,

(2) f has a modulus of uniform continuity, and

(3) there is an integer m and sequences $\langle n_k \rangle_{n \in \mathbb{N}}$ and $\langle \langle B_{k,i} \rangle_{i < m_k} \rangle_{k \in \mathbb{N}}$ such that for each k, $m_k < m$, each $B_{k,i}$ is an open ball of radius at most 2^{-k} contained in exactly one ball in the list $\langle B_{k-1,i} \rangle_{i < m_{k-1}}$, and for every rational point z exterior to $\cup_{i < m_k} B_{k,i}$ we have $|f(z) - z| > 2^{-n_k}$.

Then f has a fixed point in I^2.

What does (3) mean?

Except at m spots, $|f(z) - z|$ is nicely bounded away from 0.

Arnold's proof

To show that $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ has a zero, define

$$R = 2 + |a_1| + \cdots + |a_n|, \text{ and}$$

$$g(z) = \begin{cases} z - f(z)/(Re^{i(n-1)\theta r} & \text{for } |z| \leq 1 \\ z - f(z)/(Rz^{n-1}) & \text{for } |z| > 1. \end{cases}$$

g(z) is continuous and maps the closed disk of radius R into itself.

By Brouwer's fixed point theorem, $g(z)$ has a fixed point, which is also a zero of $f(z)$.

Bibliography

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3. N. Shioji and K. Tanaka, *Fixed point theory in weak second-order arithmetic,* Annals of Pure and Applied Logic, **47** (1990), pp. 167–188.

4. S. G. Simpson, *Subsystems of Second Order Arithmetic*, Springer-Verlag, Berlin, Heidelberg, 1999.