# Two combinatorial proofs and some related questions

Jeff Hirst[1]
Appalachian State University
Boone, NC

These slides are available at: www.mathsci.appstate.edu/~jlh

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# Proof 1: Hindman's theorem implies $ACA_0$

**Hindman's theorem** (HT) [6] If $f : \mathbb{N} \to k$ then there is a color $j$ and an infinite set $X \subset \mathbb{N}$ such that whenever $F \subset X$ is a finite set, $f(\sum F) = j$.

The following form of Hindman's theorem is provable equivalent over $RCA_0$ [1]:

(FUT) If $f : \mathbb{N}^{<\mathbb{N}} \to k$ then there is a color $j$ and an infinite increasing sequence of finite sets, $X_0 < X_1 < X_2 < \ldots$ such that whenever $F \subset \mathbb{N}$ is a finite set, $f(\cup_{i \in F} X_i) = j$.

$X_0 < X_1 < X_2 < \ldots$ means

$$\max(X_0) < \min(X_1) < \max(X_1) < \min(X_2) < \ldots$$

**Theorem:** Over $RCA_0$, FUT (and hence HT) implies $ACA_0$. (Theorem 2.2 of Blass, Hirst, and Simpson [1])

Ideas from the proof:

- Use FUT to prove that the range of an injection $g$ exists.
- Given $g$, define the coloring $f : \mathbb{N}^{<\mathbb{N}} \to 2$.
- Apply FUT to $f$ and verify that the range of $g$ can be calculated from any monochromatic sequence.

# Beginning of proof of FUT $\to$ ACA$_0$

Suppose $g : \mathbb{N} \to \mathbb{N}$ is an injection.

Given a finite set $X \subset \mathbb{N}$, define the *very short gaps* of $X$.

- Suppose $X$ is $x_0 < x_1 < x_2 < \cdots < x_n$.
- Say that $(x_i, x_{i+1})$ is a very short gap of $X$ if

$$x_i \cap g[x_{i+1}] \neq x_i \cap g[x_n]$$

Example:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $g(n)$ | 3 | 4 | 2012 | 2 | 6 | 7 | 0 | 8 | 9 | 1 |

$X$ is $\{$ 2, 5, 6, 8 $\}$

The range of $g$ on $(5, 8]$ contains an element less than 2, so $(2, 5)$ is a very short gap.

The range of $g$ on $(6, 8]$ contains no element less than 5, so $(5, 6)$ is not a very short gap.

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$$x_i \cap g[x_{i+1}] \neq x_i \cap g[x_n]$$

- Let $\mathrm{VSG}(X)$ be the cardinality of the set of very short gaps of $X$.
- Define $f(X) = \mathrm{VSG}(X) \mod 2$. ($f$ is a parity coloring.)

Apply FUT to find $\mathcal{S}$, an increasing sequence of finite sets $X_0 < X_1 < X_2 < \dots$ such that $f$ takes the same value on every finite union.

## Short gaps vs. very short gaps

$(x_i, x_{i+1})$ is a short gap if the range of $f$ on $(x_{i+1}, \infty)$ contains an element less than $x_i$.

Example revisited:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $g(n)$ | 3 | 4 | 2012 | 2 | 6 | 7 | 0 | 8 | 9 | 1 |

$X$ is $\{$ 2, 5, 6, 8 $\}$

$(2, 5)$ is a very short gap.
$(5, 6)$ is not a very short gap. $(6, 8)$ is not a very short gap.

Because $g(9) = 1$, $(2, 5)$, $(5, 6)$ and $(6, 8)$ are short gaps.

$(6, 10)$ might or might not be a short gap of $\{2, 6, 10\}$.

# SG vs. VSG

$(x_i, x_{i+1})$ is a short gap if the range of $f$ on $(x_{i+1}, \infty)$ contains an element less than $x_i$.

Example revisited:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
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SG$(X)$ is shorthand for the number of short gaps of $X$.

## The monochromatic set encodes information about the short gaps

Recall: $f$ takes the same value on every finite union of elements of $\mathcal{S} = \langle X_0, X_1, \ldots \rangle$. (Parity of VSG.)

**Claim:** If $F$ is a finite union of elements of $\mathcal{S}$, then $\mathrm{SG}(F)$ is even.
Proof: Fix $F$. Pick $n$ so big that no value less than $\max(F)$ appears in the range of $g$ on $(\min(X_n), \infty)$. Consider $F \cup X_n$.

- The short gaps of $F$ are also very short gaps of $F \cup X_n$.
- The very short gaps of $X_n$ are also very short gaps of $F \cup X_n$.
- $(\max F, \min X_n)$ is not a very short gap $F \cup X_n$.
- Summarizing: $\mathrm{VSG}(F \cup X_n) = \mathrm{SG}(F) + \mathrm{VSG}(X_n)$

Since $\mathrm{VSG}(F \cup X_n) = \mathrm{VSG}(X_n) \mod 2$, $\mathrm{SG}(F)$ is even.

# End of the proof that FUT implies $ACA_0$

**Claim:** If $F$ is a finite union of elements of $\mathcal{S}$ and $X \in \mathcal{S}$ satisfies $F < X$, then $(\max F, \min X)$ is not a short gap.

Proof: Visualize ____F____$(\max F, \min X)$____X____.

$\text{SG}(F \cup X_n) = \text{SG}(F) + \text{SG}(\{\max F, \min X\}) + \text{SG}(X)$ so

$0 = 0 + \text{SG}(\{\max F, \min X\}) + 0 \mod 2.$

**Claim:** The range of $g$ is computable from $\mathcal{S}$.

Proof: $\exists t(g(t) = n)$ iff $\exists t < \min X_{n+1}\ (g(t) = n)$

# Proof 2: $\mathrm{RT}^2_2$ implies the Free Set Theorem for pairs

Free Set Theorem for pairs ($\mathrm{FS}(2)$): If $f : [\mathbb{N}]^2 \to \mathbb{N}$ then there is an infinite set $X \subset \mathbb{N}$ such that for all $(i, j) \in [X]^2$, if $f(i, j) \in X$ then $f(i, j) = i$ or $f(i, j) = j$.

**Theorem:** $\mathrm{RCA}_0$ plus Ramsey's theorem for pairs and two colors ($\mathrm{RT}^2_2$) proves $\mathrm{FS}(2)$. (Appears in Cholak, Guisto, Hirst, and Jockusch [2].)

Proof: Suppose $f : [\mathbb{N}]^2 \to \mathbb{N}$ and assume $\mathrm{RT}^2_2$. We can use $\mathrm{RT}^2_5$, if we like.

Define $g : [\mathbb{N}]^2 \to 5$ by the formula $g(i,j) = \begin{cases} 0 & \text{if } f(i,j) < i \\ 1 & \text{if } f(i,j) = i \\ 2 & \text{if } f(i,j) \in (i,j) \\ 3 & \text{if } f(i,j) = j \\ 4 & \text{if } j < f(i,j) \end{cases}$

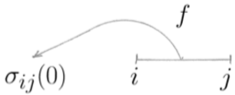
and let $M$ be an infinite set that is monochromatic for $g$.

- $g([M]^2) = 1$ implies $\mathrm{FS}(2)$ for $M$, since $f(i,j)$ is always $i$.
- $g([M]^2) = 3$ implies $\mathrm{FS}(2)$ for $M$, since $f(i,j)$ is always $j$.
- If $g([M]^2) = 4$, define $N \subset M$ by setting:
  $n_0 = m_0$, $n_1 = m_1$, and $n_{i+1}$ is the least element of $M$ greater than $\max\{f(n_j, n_k) \mid 0 \leqslant j < k \leqslant i\}$.
  $N$ satisfies $\mathrm{FS}(2)$.

• A more challenging case:

$$g([M]^2) = 0. \text{ (So } f(i,j) < i \text{ for all } i,j \in M.\text{)}$$

For each $i$ and $j$ define the sequence $\sigma_{ij}$ as follows:



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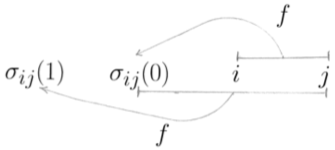



Continue as long as the sequence decreases.

Define

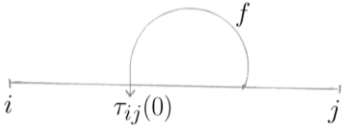
$h : [M]^2 \to 2$ by letting $h(i,j)$ be the parity of the length of $\sigma_{ij}$.

Apply $RT^2_2$ and find an infinite monochromatic set $X$ for $h$.

If $i$, $j$, and $f(i,j)$ are all in $X$, then $\sigma_{ij} = f(i,j)^\frown\sigma_{f(i,j),j}$, contradicting that their lengths have the same parity. Thus, if $i,j \in X$, then $f(i,j) \notin X$, so $X$ satisfies $FS(2)$.

# End of the second proof

- The last case: $g([M]^2 = 2$. (So $i < f(i,j) < j$ for all $i, j \in M$.)
As in the previous case, for each $i$ and $j$ define a sequence:



# End of the second proof

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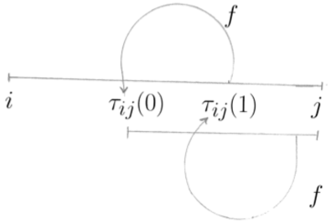


Repeat as long as the sequence increases. (It's bounded above by $j$.)

Let $h(i,j)$ be the parity of the length of $\tau_{ij}$, and argue as before.

# Related questions – FUT

Suppose $f : \mathbb{N}^{<\mathbb{N}} \to r$. Say that

$f$ is stable-t if for every $t$ there is a $b$ such that $f$ is constant on all sets containing $t$ and meeting $(b, \infty)$.

$f$ is stable-c if for every $t$ there is a $b$ such that $f$ is constant on all sets containing $t$ and of size at least $b$.

Conj: $RCA_0$ proves that FUT for stable-t colorings is equivalent to polarized Ramsey's Theorem for pairs.

Conj: $ACA_0$ proves FUT for stable-c colorings.

Does stable-c FUT imply $ACA_0$?

Is there a related version of COH?

# Related Questions – FUT plus thin/free set

Thin set for unions (TSU): Suppose $f : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$. Then there is a infinite increasing sequence of finite sets, $X_0 < X_1 < X_2 < \dots$ such that $\{f(\cup_{i \in F} X_i) \mid F \in \mathbb{N}^{<\mathbb{N}}\} \neq \mathbb{N}$.

There is also a free set theorem for unions FSU (see [2]).

Exer: $\mathrm{RCA}_0$ proves HT $\to$ TSU.

Does TSU imply anything?

Prop: Milliken's theorem for triples implies FSU. [2]

Does HT imply FSU?

Does Milliken's theorem for triples imply the thin set theorem for all $k$?

Does the polarized thin set theorem imply anything?
How about the polarized free set theorem?

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