Counting uses of Ramsey's theorem

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Motivating question:

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Can you prove $\mathrm{RT}(2,4)$ with one use of $\mathrm{RT}(2,2)$?

$\mathrm{RT}(2,n)$ is Ramsey's theorem for pairs and n colors:

Given $f : ([\mathbb{N}]^2) \to n$, we can find an infinite $X \subset \mathbb{N}$ and an $i < n$ such that $f([X]^2) = i$.

Vocabulary:

f is a coloring with n colors.

X is an infinite monochromatic set.

Further motivation

RCA_0 can prove $\mathsf{RT}(2,4)$ with two uses of $\mathsf{RT}(2,2)$.

Given $f : [\mathbb{N}]^2 \to 4$, define:

$$g_1(m,n) = \begin{cases} 0 & \text{if } f(m,n) \in \{0,1\} \\ 1 & \text{if } f(m,n) \in \{2,3\} \end{cases}$$

Let $X = \{x_0, x_1, \ldots\}$ be an infinite monochromatic set for g_1.
Note that $f([X]^2) \in \{0,1\}$ or $f([X]^2) \in \{2,3\}$.

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Note that $f([X]^2) \in \{0,1\}$ or $f([X]^2) \in \{2,3\}$. Define:

$$g_2(m,n) = \begin{cases} 0 & \text{if } f(x_m,x_n) \text{ is even} \\ 1 & \text{if } f(x_m,x_n) \text{ is odd} \end{cases}$$

Let Y be an infinite monochromatic set for g_2. Then
$Z = \{x_m \mid m \in Y\}$ is monochromatic for f.

Motivating question:

Can you prove $\mathsf{RT}(2,4)$ with one use of $\mathsf{RT}(2,2)$?

Answers:

Motivating question:

Can you prove $\mathrm{RT}(2,4)$ with one use of $\mathrm{RT}(2,2)$?

Answers:

NO

YES

Motivating question:

Can you prove $\mathrm{RT}(2,4)$ with one use of $\mathrm{RT}(2,2)$?

Answers:

NO: The intuitionistic system $i\mathrm{RCA}_0^\omega$ cannot prove $\mathrm{RT}(2,4)$ with one typical use of $\mathrm{RT}(2,2)$.

YES: RCA_0 can prove $\mathrm{RT}(2,4)$ with one use of $\mathrm{RT}(2,2)$.

One is not enough: Vocabulary

An axiom system formulated by Kohlenbach [3]

$i\mathrm{RCA}_0^\omega$ includes:

$\widehat{\text{E-HA}}^\omega_\upharpoonright$	Intuitionistic arithmetic in all finite types with restricted induction and primitive recursion
$\text{QF-AC}^{1,0}$	A choice scheme that implies the recursive comprehension axiom (RCA)

We consider *problems* of the form $\mathrm{P} : \forall x(p_1(x) \to \exists y\, p_2(x, y))$

$p_1(x)$	means x is an instance of the problem P
$p_2(x, y)$	means y is a solution of the instance x of the problem P.

One is not enough: Formalized Weihrauch reductions

Given problems:
P : $\forall x(p_1(x) \to \exists y\, p_2(x, y))$ and Q : $\forall x(q_1(x) \to \exists y\, q_2(x, y))$

in the language of $i\mathrm{RCA}_0^\omega$, we use $\mathrm{Q} \leqslant_W \mathrm{P}$ to abbreviate

$$\exists \varphi \exists \psi \forall u\, (q_1(u) \to (p_1(\varphi(u)) \land \forall y[p_2(\varphi(u), y) \to q_2(u, \psi(u, y))]))$$

Which says that there are functionals φ and ψ such that

$q_1(u)$	If u is an instance of Q
$p_1(\varphi(u))$	then $\varphi(u)$ is an instance of P
$p_2(\varphi(u), y)$	such that whenever y is a solution the instance $\varphi(u)$ of the problem P
$q_2(u, \psi(u, y))$	$\psi(u, y)$ computes a solution to the instance u of the problem Q

In computability theory, φ and ψ are computable functionals

One is not enough: What does one mean?

A theory proves Q with one typical use of P if

- From $q_1(u)$ we can deduce the existence of x_u, an instance of P.
- From $p_1(x_u) \rightarrow \exists y p_2(x_u, y)$ we can deduce the existence of $v_{x_u,y}$ with $q_2(u, v_{x_u,y})$.

Theorem: If P and Q are nice problems then $i\mathrm{RCA}_0^\omega$ proves Q with one typical use of P if and only if $i\mathrm{RCA}_0^\omega \vdash \mathrm{Q} \leqslant_W \mathrm{P}$.

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Nice problems: $q_1(u) \to (p_1(x) \wedge [p_2(x, y) \to q_2(u, v)])$ is in Γ_1

Proof uses modified reducibility [2], adapted from Kohlenbach.

Rutger Kuyper [4] has proved related results.

One is not enough: The main result

Theorem:
$i\mathrm{RCA}_0^\omega$ cannot prove $\mathrm{RT}(2,4)$ with one typical use of $\mathrm{RT}(2,2)$.

Proof in three steps:

1: $\mathrm{RT}(2,4) \not\leqslant_W \mathrm{RT}(2,2)$ Shafer et al [1]

2: $i\mathrm{RCA}_0^\omega \not\vdash \mathrm{RT}(2,4) \leqslant_W \mathrm{RT}(2,2)$

$i\mathrm{RCA}_0^\omega$ lies only by omission

3: Apply the previous result.

Sometimes, one is enough

Claim: RCA_0 can prove $\mathrm{RT}(2,4)$ with one use of $\mathrm{RT}(2,2)$.

Sketch: Given $f : [\mathbb{N}]^2 \to 4$, either

- $\exists X$ infinite with $f([X]^2) \subset \{a_0, a_1\} \subset \{0, 1, 2, 3\}$

  or
- there is no such X.

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  Fix such an $X = \{x_0, x_1, \ldots\}$ and set $j = 0$
- there is no such X.

  Let $X = \mathbb{N}$ and set $j = 1$

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Define $g : [\mathbb{N}]^2 \to 2$ by:

$$g(m,n) = \begin{cases} 0 & \text{if } j = 0 \wedge f(x_m, x_n) = a_0 \\ 1 & \text{if } j = 0 \wedge f(x_m, x_n) = a_1 \\ 0 & \text{if } j = 1 \wedge f(x_m, x_n) \in \{0,1\} \\ 1 & \text{if } j = 1 \wedge f(x_m, x_n) \in \{2,3\} \end{cases}$$

If Y is mono. for g, then $\{x_m \mid m \in Y\}$ is mono. for f

Some references

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[2] Jeffry L. Hirst and Carl Mummert, *Reverse mathematics and uniformity in proofs without excluded middle*, Notre Dame J. Form. Log. **52** (2011), no. 2, 149–162. DOI 10.1215/00294527-1306163.

[3] Ulrich Kohlenbach, *Higher order reverse mathematics*, Reverse Mathematics 2001 (Stephen G. Simpson, ed.), Lect. Notes Log., vol. 21, Assoc. Symbol. Logic, La Jolla, CA, 2005, pp. 281–295.

[4] Rutger Kuyper, *On Weihrauch reducibility and intuitionistic reverse mathematics*. Preprint, arXiv:1511.05189v1.