Counting, Graphs, and Bells

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Change ringing is the art of sequentially playing all possible permutations of a set of bells.

Example: An extent on 2 bells:

1 2 row: list of bells

change / permutation (

2 1

(

1 2

An *extent* is a composition that

- begins with rounds (bells played in decreasing pitch),
- contains one row for each possible permutation, and
- finishes with rounds.

Not all changes are legal

Each change must be product of disjoint transpositions of adjacent bells.

For three bells:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \text{are legal}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{are not legal}$$

For four bells:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad \text{is legal}$$

There are exactly two extents on three bells.

123	
213	132
231	312
321	321
312	231
132	213
123	

$$\circ a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad \circ\, b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

The unpleasant detail: Generally, rows do not equal the lower line of their permutation.

$$123$$

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad |$$

$$213$$

$$b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \qquad |$$

$$231$$

$$ba(1) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} (1) = 3$$

$$ba(2) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} (2) = 1$$

$$ba(3) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} (3) = 2$$

$$\text{so, } ba = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

However, after ba, 1 goes in the 3^{rd} slot, etc.

The strange resolution:

$$ba = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$(ba)^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

In general, the row resulting from the change $a_n a_{n-1} \ldots a_2 a_1$ is the second line of the permutation $(a_n a_{n-1} \ldots a_2 a_1)^{-1}$.

Also, every legal change is its own inverse, so

$$a_1 a_2 \ldots a_{n-1} a_n a_n a_{n-1} \ldots a_2 a_1 = e.$$

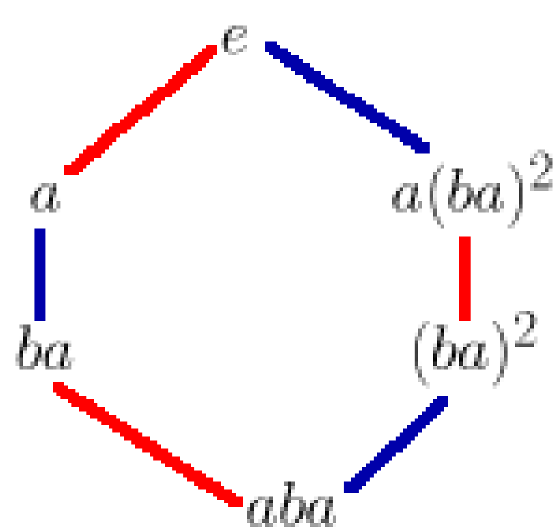
That is, $(a_n a_{n-1} \ldots a_2 a_1)^{-1} = a_1 a_2 \ldots a_{n-1} a_n$.

$$ab = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

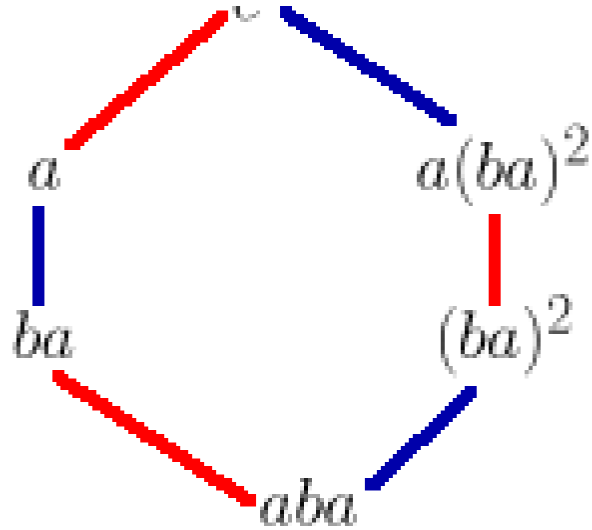
Cayley Diagrams

The set of permutations of three things forms a group. It's S_3, the symmetric group for $n = 3$. S_3 has 3! elements, and is generated by the permutations a and b.

$$S_3 = \{e, a, ba, aba, (ba)^2, a(ba)^2\}$$



$$a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$



$$a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

- The extents on three bells are the Hamilton paths through the Cayley diagram, starting at e.
- The extents can also be represented by identity words, namely *ababab* and *bababa*.
- These identity words are unusual in that they are of length 6 (the order of S_3) and contain no identity subwords.

Plain Bob Minimus:
A traditional extent on 4 bells

The rows:

1234	1342	1423
2143	3124	4132
2413	3214	4312
4231	2341	3421
4321	2431	3241
3412	4213	2314
3142	4123	2134
1324	1432	1243
		1234

The word:

$$(cabababa)^3 = e, \text{ where}$$

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$

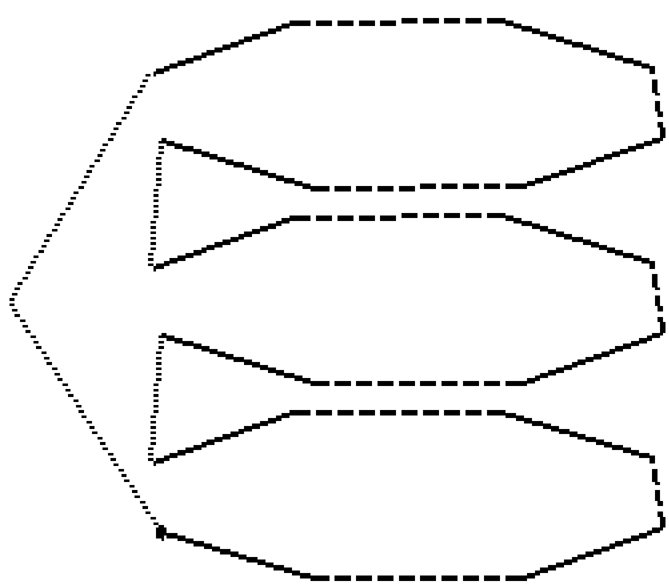
Plain Bob Minimus:
A traditional extent on 4 bells

The word:

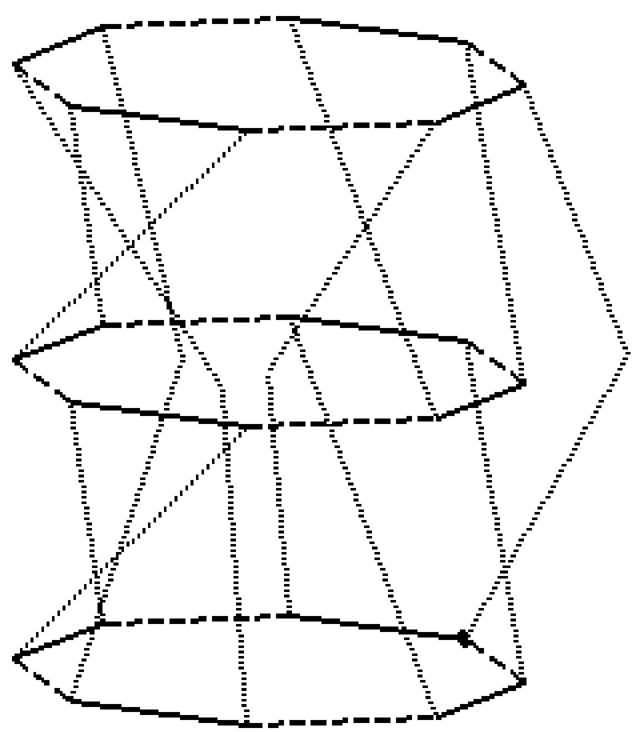
$$(cabababa)^3 = e, \text{ where}$$

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$

The Hamilton path through the Cayley diagram:



The Cayley diagram for S_4
(generated by a, b and c)



References

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