Graphs, Free Sets, and Reverse Mathematics

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Reverse Mathematics

Goal: Determine what set existence axioms are needed to prove familiar theorems.

Method: Prove results of the form

$$\mathbf{RCA_0} \vdash \mathbf{AX} \leftrightarrow \mathbf{THM}$$

where:

- $\mathbf{RCA_0}$ is a weak axiom system,
- **AX** is a set existence axiom selected from a small hierarchy of axioms, and
- **THM** is a familiar theorem.

$\mathbf{RCA_0}$

Recursive Comprehension

Language:

x, y, z variables representing integers
X, Y, Z variables representing sets of integers
0, 1, $+$, $\times$, $=$, $<$, and $\in$

Axioms:

basic arithmetic axioms
(0, 1, $+$, $\times$, $=$, and $<$ behave as usual.)

Restricted induction
$(\psi(0) \wedge \forall n(\psi(n) \rightarrow \psi(n+1))) \rightarrow \forall n \psi(n)$
where $\psi(n)$ has (at most) one x quantifier.

Recursive set comprehension
If $\theta \in \Sigma_1^0$ and $\psi \in \Pi_1^0$, and
$\forall n(\theta(n) \leftrightarrow \psi(n))$, then there is
a set X such that $\forall n(n \in X \leftrightarrow \theta(n))$

What can $\mathbf{RCA_0}$ prove?

Arithmetic needed for coding.

Lots of finite graph theory, e.g.

Thm ($\mathbf{RCA_0}$) Every finite graph with no odd cycles is bipartite.

A little analysis, e.g.

Thm ($\mathbf{RCA_0}$) If $\langle I_n \rangle_{n \in \mathbb{N}}$ is a sequence of nested real intervals, then there is a real number in their intersection.

Weak König's Lemma

Statement: Big 0-1 trees are tall.

More formally: If T is an infinite tree in which each node is labeled 0 or 1, then T contains an infinite path.

$\mathbf{WKL_0}$ is $\mathbf{RCA_0}$ plus Weak König's Lemma.

Note: $\mathbf{RCA_0} \nvdash \mathbf{WKL_0}$

Some reverse mathematics!

Thm ($\mathbf{RCA_0}$) The following are equivalent:

1) $\mathbf{WKL_0}$.

2) Every graph with no cycles of odd length is bipartite.

Proof: To prove that 1) $\rightarrow$ 2), we should 2-color the nodes of an arbitrary graph with no odd cycles by using a tree.

The reversal
Proof that "bipartite thm" implies $\mathbf{WKL_0}$

A reversal tool:

Thm ($\mathbf{RCA_0}$) T.F.A.E.:

1) $\mathbf{WKL_0}$

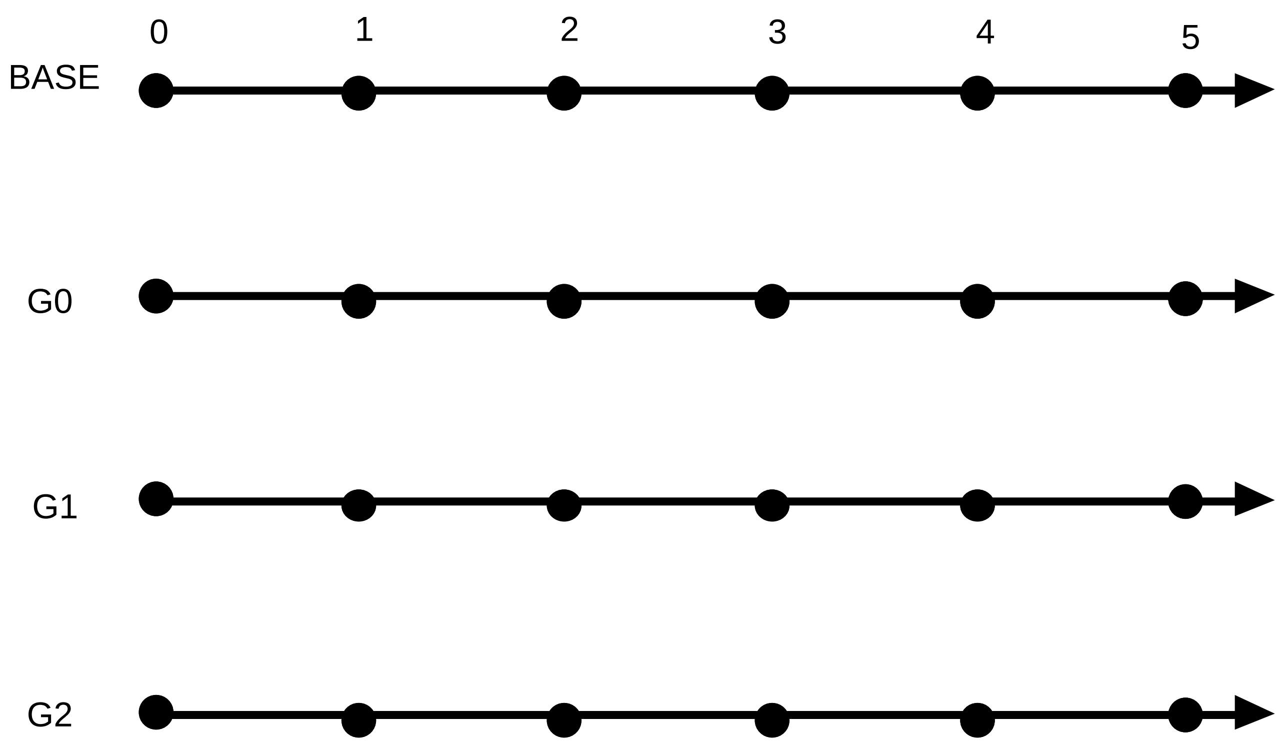
2) If f and g are 1-1 functions from $\mathbb{N}$ into $\mathbb{N}$ and $Ran(f) \cap Ran(g) = \emptyset$, then there is a set X such that $Ran(f) \subset X$ and $X \cap Ran(g) = \emptyset$.

Sketch of the reversal: Use a 2-coloring of a graph with no odd cycles to separate the ranges of some arbitrary functions.

Sample construction: Suppose we are given f and g such that $\mathbb{N}$ and $Ran(f) \cap Ran(g) = \emptyset$.

If, for example, $f(3) = 0$ and $g(4) = 2$, we will construct the graph G as follows:

Associate straight links with f
Associate shifted links with g



Related Computability Results

(Bean) There is a computable graph with no cycles of odd length that has no computable 2-coloring.

(Bean) Every computable graph with no cycles of odd length has a low 2-coloring.

Arithmetical Comprehension

$\mathbf{ACA_0}$ consists of $\mathbf{RCA_0}$ plus the following arithmetical comprehension scheme:

For any formula $\theta(n)$ with only number quantifiers, the set $\{n \in \mathbb{N} \mid \theta(n)\}$ exists.

Note: $\mathbf{WKL_0} \nvdash \mathbf{ACA_0}$, but $\mathbf{ACA_0} \vdash \mathbf{WKL_0}$

A reversal tool:

Thm ($\mathbf{RCA_0}$) T.F.A.E.:

1) $\mathbf{ACA_0}$

2) If $f : \mathbb{N} \to \mathbb{N}$ is 1-1, then $Ran(f)$ exists.

$\mathbf{ACA_0}$ and Graph Theory

Thm ($\mathbf{RCA_0}$) T.F.A.E.:

1) $\mathbf{ACA_0}$

2) $\mathbf{RT}(3,2)$

Ramsey's theorem for triples and two colors: Given $g : [\mathbb{N}]^3 \to \{0,1\}$ there is an infinite set $H \subset \mathbb{N}$ such that g is constant on $[H]^3$.

Sketch of the reversal

We will use Ramsey's theorem to define the range of an arbitrary function. Suppose we want to find the range of the function f. Define $g : [\mathbb{N}]^3 \to \{0,1\}$ by:

$$g(x,y,z) = \begin{cases} 0 \text{ if } \exists j \in [y,z](f(j) \le x) \\ 1 \text{ otherwise} \end{cases}$$

The range of f can be computed from any infinite homogeneous set for g.

Related computability results

Every computable two coloring of triples has an arithmetically definable infinite homogeneous set.

There is a computable two coloring of triples f such that $\mathbf{0}'$ is computable from every infinite set that is homogeneous for f.

(These follow easily from work of Jockusch.)

Arithmetical Transfinite Recursion

$\mathbf{ATR_0}$ consists of $\mathbf{RCA_0}$ plus axioms that allow iteration of arithmetical comprehension along any well ordering. This allows transfinite constructions.

A reversal tool:

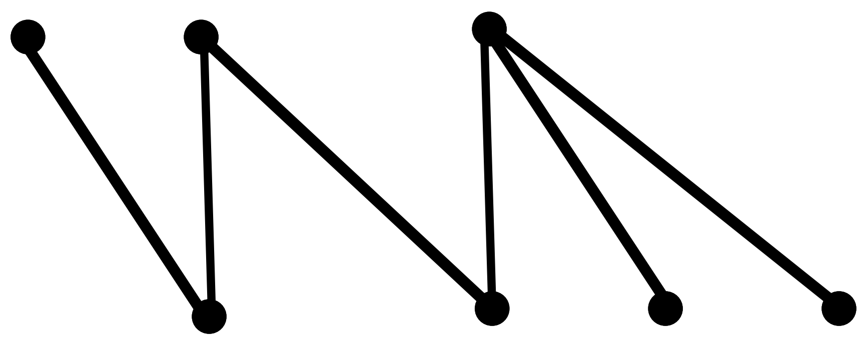
Thm ($\mathbf{RCA_0}$) T.F.A.E.:

1) $\mathbf{ATR_0}$

2) If α and β are well orderings, then $\alpha \leq \beta$ or $\beta \leq \alpha$.

$\mathbf{ATR_0}$ and Graph Theory

(C, M) is a König cover for the graph G if C contains at least one vertex from each edge of G, M is a collection of disjoint edges, and C consists of exactly one vertex from each edge of M.



Thm ($\mathbf{RCA_0}$) T.F.A.E.:

1) $\mathbf{ATR_0}$

2) Countable König's Duality Theorem: Every countable bipartite graph has a König cover.

Proof of 2) $\rightarrow$ 1): Aharoni, Magidor, and Shore
Proof of 1) $\rightarrow$ 2): Simpson

Π^1_1 comprehension

The system $\mathbf{\Pi^1_1 - CA_0}$ consists of $\mathbf{RCA_0}$ and the axioms asserting the existence of the set $\{n \in \mathbb{N} \mid \theta(n)\}$ for $\theta \in \Pi^1_1$. (That is, θ has one universal set quantifier and no other set quantifiers.)

A reversal tool followed by graph theory:

Thm ($\mathbf{RCA_0}$) T.F.A.E.:

1) $\mathbf{\Pi^1_1 - CA_0}$

2) If $\langle T_i \rangle_{n\in\mathbb{N}}$ is a sequence of trees then there is a function $f : \mathbb{N} \to \{0,1\}$ such that $f(n) = 1$ iff T_n is well founded.

3) For any graph H, and any sequence of graphs $\langle G_i \rangle_{i\in\mathbb{N}}$, there is a function $f : \mathbb{N} \to \{0,1\}$ such that $f(n) = 1$ iff H is isomorphic to a subgraph of G. (Hirst and Lempp)

Free sets and thin sets

Recently, Friedman has introduced the following combinatorial statements:

FS(n) (Free set theorem): If $f : [\mathbb{N}]^n \to \mathbb{N}$, then there is an infinite set H such that for all $\vec{x} \in [H]^n$,

$$\text{if } f(\vec{x}) \in H \text{ then } f(\vec{x}) \in \vec{x}.$$

TS(n) (Thin set theorem): If $f : [\mathbb{N}]^n \to \mathbb{N}$, then there is an infinite set H such that

$$f([H]^n) \neq \mathbb{N}.$$

Results on $\mathbf{FS}(n)$ and $\mathbf{TS}(n)$

$$\forall n \in \omega\ \mathbf{ACA_0} \vdash \mathbf{FS}(n)$$

$$\mathbf{ACA_0} \nvdash \forall n \mathbf{FS}(n)$$

$$\mathbf{RCA_0} \vdash \forall n (\mathbf{FS}(n) \rightarrow \mathbf{TS}(n))$$

$$\mathbf{RCA_0} \vdash \mathbf{RT}(2,2) \rightarrow \mathbf{FS}(2)$$

$$\mathbf{RCA_0} + \mathbf{FS}(2) \nvdash \mathbf{ACA_0}$$

$$\mathbf{WKL_0} \nvdash \mathbf{TS}(2)$$

Does $\mathbf{RCA_0} \vdash \mathbf{TS}(3) \rightarrow \mathbf{ACA_0}$?

Does $\mathbf{RCA_0} \vdash \mathbf{TS}(3) \rightarrow \mathbf{RT}(1)$?

A few references

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