Milliken's Theorem, Ultrafilters, and Reverse Mathematics

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Reverse mathematics can measure the proof-theoretic and computability-theoretic strength of theorems.

Subsystems of second order arithmetic:

RCA_0: (recursive comprehension)

- ordered semi-ring axioms
- induction for Σ^0_1 formulas
- existence axioms for relatively computable sets
- model: ω and the computable sets

ACA_0: (arithmetical comprehension)

- RCA_0 plus existence axioms for sets defined by formulas containing number quantifiers (no set quantifiers)
- model: ω and the arithmetically definable sets

$\mathsf{FS}(X) :=$ all sums of finite subsets of X

Example: Suppose $X = \{1, 2, 5\}$.

Then $\mathsf{FS}(X) = \{1, 2, 3, 5, 6, 7, 8\}$

No repeating!

Theorem 1 (Hindman's Theorem [3]). *Given $G \subseteq \mathbb{N}$, there is an infinite set $X \subseteq \mathbb{N}$ such that $\mathsf{FS}(X) \subseteq G$ or $\mathsf{FS}(X) \subseteq G^c$.*

Example: Suppose G is the set of natural numbers that have an even number of factors of 2 in their prime factorization.

0	1	2	3	4	5	6	7	8	16
G	G		G	G	G		G		G

Candidates for X:

1. G doesn't work, because $3 + 5 = 8$.
2. G^c doesn't work, because $2 + 6 + 8 = 16$.
3. $X = \{2^1, 2^3, 2^5, 2^7, \dots\}$ works! Every nonrepeating finite sum is in G^c.

Sometimes it's hard to find X.

Hindman's Theorem and Reverse Math

Theorem 2. (Blass, Hirst, Simpson [1]) (RCA_0) *Hindman's theorem proves* ACA_0.

ACA_0^+ consists of ACA_0 plus the existence of the ω jump of each set.

Theorem 3. (Blass, Hirst, Simpson [1]) ACA_0^+ *proves Hindman's theorem.*

Theorem 4. (Blass, Hirst, Simpson [1]) ACA_0^+ *proves this iterated version of Hindman's theorem: Given* $\langle G_i \rangle_{i \in \mathbb{N}}$ *there is an increasing sequence* $\langle x_i \rangle_{i \in \mathbb{N}}$ *such that for each value of* j*, either* $\mathsf{FS}(\{x_i \mid i > j\}) \subseteq G_j$ *or* $\mathsf{FS}(\{x_i \mid i > j\}) \subseteq G_j^c$.

Downward Translations

For $X \subseteq \mathbb{N}$ and $m \in \mathbb{N}$, let

$$X - m = \{y \in \mathbb{N} \mid y + m \in X\}$$

n	0	1	2	3	4	5	6	7	8
$n \in X$	0	1	1	0	0	0	1	1	1
$n \in X - 2$	1	0	0	0	1	1	1	?	?

A countable field of sets (Boolean algebra of sets) is a collection of subsets of $\mathbb{N}$ which is closed under intersection, (finite) union, and complement.

A downward translation algebra is a field of sets that is closed under downward translations.

RCA_0 can prove that the downward translation algebra generated by $\langle G_i \rangle_{i \in \mathbb{N}}$ exists.

Ultrafilters

An ultrafilter on a field of sets F is a subset $U \subseteq F$ satisfying:

1. $\emptyset \notin U$
2. if $X_1, X_2 \in U$ then $X_1 \cap X_2 \in U$
   (closed under intersections)
3. $\forall X \in U\ \forall Y \in F\ (X \subseteq Y \rightarrow Y \in U)$
   (closed under supersets)
4. $\forall X \in F\ (X \in U \vee X^c \in U)$

An ultrafilter U is
almost downward translation invariant if

$$\forall X \in U\ \exists x \in X\ (x \neq 0 \wedge X - x \in U)$$

Theorem 5. (Hindman [2]) *Assuming CH, Hindman's theorem holds if and only if there is an almost downward translation invariant ultrafilter on the field of subsets of* $\mathbb{N}$.

P-theorem 6. (RCA_0) *The following are equivalent:*

1. *The iterated version of Hindman's Theorem.*
2. *Every countable downward translation algebra has an almost downward translation invariant ultrafilter.*

Ideas from the proof of P-theorem 6

- IHT $\to$ ultrafilters

Enumerate the d.t. algebra, $\langle G_i \rangle$; apply IHT.

For each i, let $\widehat{G}_i$ denote the element of $\{G_i, G_i^c\}$ containing the homogeneous set.

$U = \{\widehat{G}_i \mid i \in \mathbb{N}\}$ is the desired u.f.

- Ultrafilters $\to$ IHT

Generate a d.t. algebra from the partitions: $G = \langle \{G_i \mid i \in \mathbb{N}\} \rangle$

Find an ultrafilter U on G.

Let $X_0 = \widehat{G}_0$ and $x_0 = 0$.

Let x_{n+1} be the least element of X_n with $x_{n+1} > x_n$ and $X_n - x_{n+1} \in U$ and let $X_{n+1} = X_n \cap (X_n - x_{n+1}) \cap \widehat{G}_{n+1}$

$X = \{x_j \mid j > 0\}$ is the desired homogeneous set.

Why is Milliken's Theorem in the title of this talk?

Notation:

$A < B$ means $\max A < \min B$

$\sum A$ abbreviates $\sum_{x \in A} x$

$\mathsf{FS3}(X)$ is the triples of increasing sums: $(\sum A, \sum B, \sum C)$ where

$$A < B < C \text{ and } A \cup B \cup C \subseteq X$$

$\mathsf{FS}n(x)$ is defined analogously

P-theorem 7. *For each standard natural number $n \geq 3$, RCA_0 can prove that the following are equivalent:*

(1) *Milliken's theorem for n-tuples: If $f : [\mathbb{N}]^n \to k$ then there is an increasing sequence $X = \langle x_i \rangle_{i \in \mathbb{N}}$ such that f is constant on $\mathsf{FS}n(X)$.*

(2) *Iterated Hindman's Theorem.*

Questions

Does ACA_0 prove Milliken's Theorem?

Does ACA_0 prove the iterated Hindman's Theorem?

Does ACA_0 prove Hindman's Theorem?

Can techniques from topological semigroups be applied to the space of ultrafilters on countable downward translation algebras to carry out proofs of combinatorial statements in ACA_0?

References

[1] A. Blass, J. Hirst, and S. Simpson. Logical Analysis of Some Theorems of Combinatorics and Topological Dynamics. In: *Logic and Combinatorics* (Editor: S. Simpson), Contemporary Mathematics, **65**:125–156, 1987.

[2] N. Hindman. The existence of certain ultrafilters on $\mathbb{N}$ and a conjecture of Graham and Rothschild. *Proc. Amer. Math. Soc.*, **36**:341–346, 1972.

[3] N. Hindman. Finite sums from sequences within cells of a partition of $\mathbb{N}$. *J. Combin. Theory Ser. A*, **17**:1–11, 1974.

[4] Keith R. Milliken. Ramsey's theorem with sums or unions. *J. Combinatorial Theory Ser. A*, 18:276–290, 1975.

[5] S.G. Simpson. *Subsystems of second order arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999.

For these slides and proofs of the P-theorems,
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