Variants of Ramsey's Theorem

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These slides are available at:
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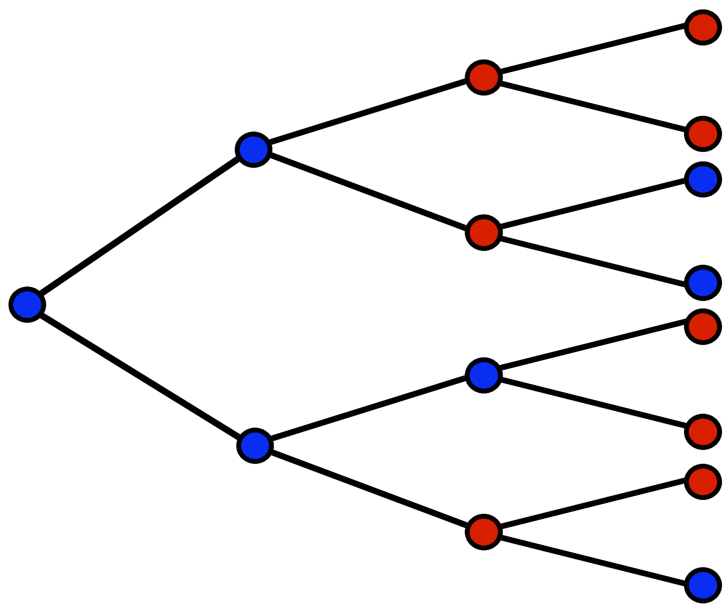
Pigeonhole principles

RT^1: If $f : \mathbb{N} \to k$ then there is a $c \leq k$ and an infinite set H such that $\forall n \in H \; f(n) = c$.

TT^1: For any finite coloring of $2^{<\mathbb{N}}$, there is a monochromatic subtree order-isomorphic to $2^{<\mathbb{N}}$.



A proof of TT^1

Lef FIN denote the set of finite subsets of $\mathbb{N}$.

Hindman's exceptionally large hammer:

Finite Union Theorem (FUT): If $f : \mathsf{FIN} \rightarrow \mathsf{k}$ then there is a $c \leq k$ and an infinite increasing sequence $\langle H_i \rangle_{i \in \mathbb{N}}$ of elements of FIN such that for every $F \in \mathsf{FIN}$

$$f(\cup_{i \in F} H_i) = c.$$

Claim: TT^1 is an easy consequence of FUT.

Logical analysis of the FUT proof of TT^1

Using results of Blass, Hirst, and Simpson (BHS)...

Computability theoretic:

Theorem (BHS) If f and $\langle H_i \rangle_{i \in \mathbb{N}}$ are as in FUT, and f is computable, then we can find $\langle H_i \rangle_{i \in \mathbb{N}} \leq 0^{(\omega)}$.

Consequence: if f is a computable coloring of $2^{<\mathbb{N}}$, then there is a monochromatic subtree computable from $0^{(\omega)}$.

Reverse mathematics:

Theorem (BHS) $\mathsf{ACA}_0^+ \vdash \mathsf{FUT}$

Consequence: $\mathsf{ACA}_0^+ \vdash \mathsf{TT}^1$

Question: Does ACA_0 prove FUT?

Another proof of TT^1

As proved in Chubb, Hirst, and McNichol (CHM)

Step 1: The proof for two colors.

Step 2: There is a least j such that there is a node τ such that at most j colors appear at or above τ.

Logical analysis

Computability theoretic: If the coloring is computable, then there is a computable monochromatic subtree.

Reverse mathematics: $\mathsf{RCA}_0 + \Sigma^0_2 - \mathsf{IND} \vdash \mathsf{TT}^1$

Jockusch: "*We do not see how to obtain the latter result starting from [the] original proof.*"

Some results on Ramsey's theorem

RT^n_k: If $f : [\mathbb{N}]^n \to k$ then there is a c and an infinite $H \subset \mathbb{N}$ such that $f([\mathbb{N}]^n) = c$.

Sample computability theoretic results

If f is computable, then there is an H which is:

- Π^0_n definable (Jockusch)
- with $H'' \leq_T 0^{(n)}$ (Cholak, Jockusch, Slaman)

There is a computable $f : [\mathbb{N}]^n \to k$ such that

- no H is computable (Specker)
- no H is Σ^0_n definable (Jockusch)
- 0^{n-2} is computable from every H (Jockusch)

Some more results on Ramsey's theorem

RT^n_k: If $f : [\mathbb{N}]^n \to k$ then there is a c and an infinite $H \subset \mathbb{N}$ such that $f([\mathbb{N}]^n) = c$.

RT^n: $\forall k \mathsf{RT}^n_k$

RT: $\forall n \mathsf{RT}^n$

Sample reverse mathematics

- For $n \geq 3$ and $k \geq 2$, $\mathsf{RCA_0} \vdash \mathsf{RT^n_k} \leftrightarrow \mathsf{ACA_0}$ (Simpson)
- $\mathsf{RCA_0} \vdash \mathsf{RT^1} \leftrightarrow \mathsf{B\Pi^0_1}$
- $\mathsf{RCA_0} \vdash \mathsf{RT} \leftrightarrow \mathsf{ACA_0'}$ (Mileti)

TT^n_k parallels RT^n_k

TT^n_k: For any k coloring of the n-tuples of comparable nodes in $2^{<\mathbb{N}}$, there is a color and a subtree order-isomorphic to $2^{<\mathbb{N}}$ in which all n-tuples of comparable nodes have the specified color.

Note: RT^n_k is an easy consequence of TT^n_k

Results in Chubb, Hirst, and McNichol:

- There is a computable coloring with no Σ^0_n monochromatic subtree. (Free.)
- Every computable coloring has a Π^0_n monochromatic subtree. (Not free.)
- For $n \geq 3$ and $k \geq 2$, $\mathsf{RCA_0} \vdash \mathsf{TT^n_k} \leftrightarrow \mathsf{ACA_0}$.

Questions about TT

Mileti showed that $\mathsf{RCA_0} \vdash \mathsf{RT} \leftrightarrow \mathsf{ACA}_0'$.

Does $\mathsf{RCA_0} \vdash \mathsf{TT} \leftrightarrow \mathsf{ACA}_0'$?

Cholak, Jockusch, and Slaman showed $\mathsf{RCA_0} + \mathsf{RT}^2_2 \nvdash \mathsf{RT}^2$.

Does $\mathsf{RCA_0} + \mathsf{TT}^2_2 \vdash \mathsf{TT}^2$?

Does $\mathsf{RCA_0} + \mathsf{TT}^2_2 \vdash \mathsf{RT}^2$?

Does $\mathsf{RCA_0} + \mathsf{TT}^2_2 \vdash \Sigma^0_2 - \mathsf{IND}$?

Polarized partitions

Preliminary results with Damir Dzhafarov:

[IPT^n_k:] If $f : [\mathbb{N}]^n \to k$ then there is a c and a sequence of infinite sets $H_1 \ldots H_n$ such that for any $x_1 < \cdots < x_n$ (with $x_i \in H_i$ for all i) we have $f(x_1 \ldots x_n) = c$.

Note: IPT^n_k is an easy consequence of RT^n_k.

Theorem: If f is computable, then there is a Π^0_n-definable $H_1 \ldots H_n$. (Free)

Theorem: There is a computable f with no Σ^0_n definable $H_1 \ldots H_n$. (Not free, but not terrible.)

Theorem: If $n \geq 3$ and $k \geq 2$, $\mathsf{RCA}_0 \vdash \mathsf{IPT}^n_k \leftrightarrow \mathsf{ACA}_0$.

Theorem: $\mathsf{RCA}_0 \vdash \mathsf{IPT} \leftrightarrow \mathsf{ACA}_0'$,

IPT^2

$f : [\mathbb{N}]^2 \to k$ is *stable* if $\lim_m f(n,m)$ exists for every n.

SRT^2 is RT^2 for stable partitions.

SIPT^2 is IPT^2 for stable partitions.

Theorem: $\mathsf{RCA}_0 \vdash \mathsf{SIPT}^2 \to \mathsf{RT}^1$

Theorem: $\mathsf{RCA}_0 \vdash \mathsf{SIPT}^2 \leftrightarrow \mathsf{SRT}^2$

Consequence: $\mathsf{RCA}_0 \vdash \mathsf{RT}^2 \to \mathsf{IPT}^2 \to \mathsf{SRT}^2$

Question: Which of the converses hold?

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