A Brief Tour of Reverse Mathematics

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Reverse Mathematics

Goal: Determine what set existence axioms are needed to prove familiar theorems.

Method: Prove results of the form

$$\mathsf{RCA}_0 \vdash \mathbf{AX} \leftrightarrow \mathbf{THM}$$

where:

- RCA_0 is a weak axiom system,
- $\mathbf{AX}$ is a set existence axiom selected from a small hierarchy of axioms, and
- $\mathbf{THM}$ is a familiar theorem.

RCA_0: Recursive Comprehension

Language:

Integer variables: x, y, z Set variables: X, Y, Z

Axioms:

basic arithmetic axioms
(0, 1, +, ×, =, and < behave as usual.)

Restricted induction
$(\psi(0) \wedge \forall n(\psi(n) \rightarrow \psi(n+1))) \rightarrow \forall n \psi(n)$
where $\psi(n)$ has (at most) one x quantifier.

Recursive set comprehension
If $\theta \in \Sigma^0_1$ and $\psi \in \Pi^0_1$, and $\forall n(\theta(n) \leftrightarrow \psi(n))$, then there is a set X such that $\forall n(n \in X \leftrightarrow \theta(n))$

What can RCA_0 prove?

Arithmetic needed for coding.

Lots of finite graph theory, e.g.

> **Thm** (RCA_0) Every finite graph with no odd cycles is bipartite.

A little analysis, e.g.

> **Thm** (RCA_0) If $\langle I_n \rangle_{n \in \mathbb{N}}$ is a sequence of nested real intervals, then there is a real number in their intersection.

Weak König's Lemma

Statement: Big very skinny trees are tall.

More formally: If T is an infinite tree in which each node is labeled 0 or 1, then T contains an infinite path.

WKL_0 is RCA_0 plus Weak König's Lemma.

Note: $\mathsf{RCA}_0 \nvdash \mathsf{WKL}_0$

Some reverse mathematics!

Thm (RCA_0) The following are equivalent:

1. WKL_0.

2. Every graph with no cycles of odd length is bipartite.

Proof: To prove that 1) $\rightarrow$ 2), we should 2-color the nodes of an arbitrary graph with no odd cycles by using a tree.

The reversal
Proof that "bipartite thm" implies WKL_0

We'll use:
Thm (RCA_0) T.F.A.E.:

1. WKL_0
2. If f and g are 1-1 functions from $\mathbb{N}$ into $\mathbb{N}$ and $Ran(f) \cap Ran(g) = \emptyset$, then there is a set X such that $Ran(f) \subset X$ and $X \cap Ran(g) = \emptyset$.

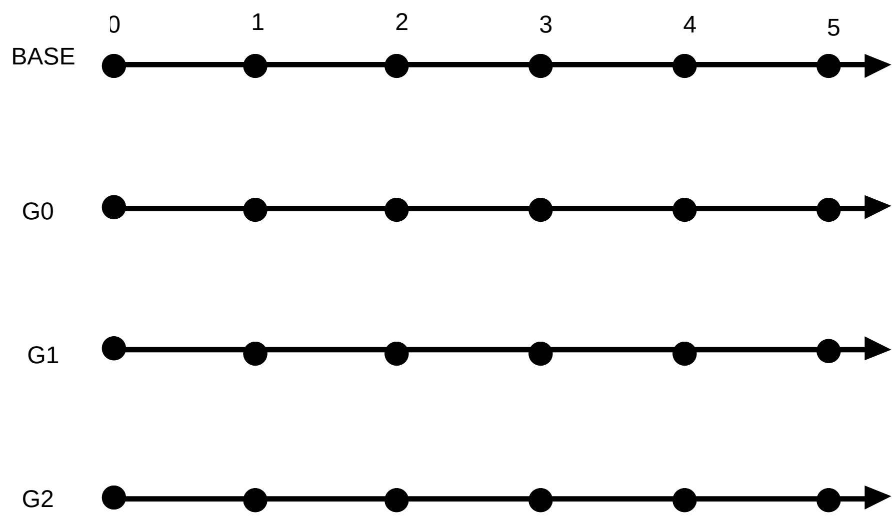
Sketch of the reversal: Use a 2-coloring of a graph with no odd cycles to separate the ranges of some arbitrary functions.

Sample construction: Suppose we are given f and g such that $\mathbb{N}$ and $Ran(f) \cap Ran(g) = \emptyset$.

If, for example, $f(3) = 0$ and $g(2) = 2$, we will construct the graph G as follows:

Associate straight links with f

Associate shifted links with g



Other theorems equivalent to WKL_0

Thm (RCA_0) T.F.A.E. (to WKL_0):

1. Every ctn. function on [0,1] is bounded. (Simpson)
2. The closed interval [0,1] is compact. (Friedman)
3. Every closed subset of $\mathbb{Q} \cap [0, 1]$ is compact. (Hirst)
4. Existence theorem for solutions to ODEs. (Simpson)
5. The line graph of a bipartite graph is bipartite. (Hirst)
6. Every countable partial order with no chains of length $k + 1$ can be decomposed into k antichains. (Hirst)

Arithmetical Comprehension

$\mathsf{ACA_0}$ is $\mathsf{RCA_0}$ plus the following comprehension scheme:

> For any formula $\theta(n)$ with only number quantifiers, the set $\{n \in \mathbb{N} \mid \theta(n)\}$ exists.

Note: $\mathsf{WKL_0} \nvdash \mathsf{ACA_0}$, but $\mathsf{ACA_0} \vdash \mathsf{WKL_0}$

The tool:

Thm ($\mathsf{RCA_0}$) T.F.A.E.:

1. $\mathsf{ACA_0}$

2. If $f : \mathbb{N} \to \mathbb{N}$ is 1-1, then $Ran(f)$ exists.

ACA_0 and Graph Theory

Thm (RCA_0) T.F.A.E.:

1. ACA_0
2. Every graph can be decomposed into its connected components.

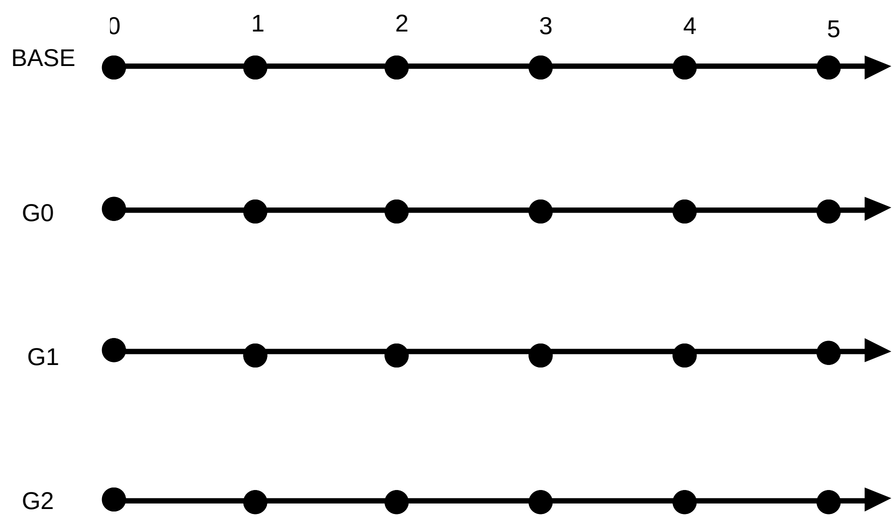
Proof: To prove that 1) implies 2), let G be a graph with vertices $v_0, v_1, ...$

Define f by letting $f(n)$ be the least j such that there is a path from v_n to v_j.

By ACA_0, f exists. f is the desired decomposition.

The reversal

Sketch: We will use a graph decomposition to define the range of an arbitrary function. Suppose we want to find the range of the function f.
Sample construction: Suppose that $f(4) = 2$ and $f(3) = 0$.



Other theorems equivalent to ACA_0

Thm (RCA_0) T.F.A.E. (to ACA_0):

1. Bolzano-Weierstraß theorem. (Friedman)
2. Cauchy sequences converge. (Simpson)
3. Every connected graph with at most one vertex of odd degree which has at least one vertex of odd or infinite degree, and which cannot be disconnected by the removal of any finite subgraph has an Euler path. (Gasarch and Hirst)
4. Arithmetical transfinite induction. (Hirst)

Arithmetical Transfinite Recursion

ATR_0 consists of RCA_0 plus axioms that allow iteration of arithmetical comprehension along any well ordering. This allows transfinite constructions.

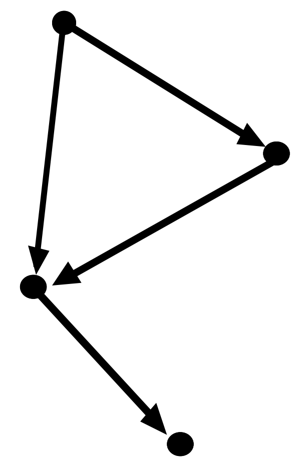
The main tool:

Thm (RCA_0) T.F.A.E.:

1. ATR_0
2. If α and β are well orderings, then $\alpha \leq \beta$ or $\beta \leq \alpha$.

ATR_0 and Graph Theory

A rank function for a directed acyclic graph is a function that maps the vertices into a well ordering, preserving the ordering induced by the edges in a nice way.



Thm (RCA_0) T.F.A.E.:

1. ATR_0
2. Every well founded directed acyclic graph with a source node has a derived sorting.

Other theorems equivalent to ATR_0

Thm (RCA_0) T.F.A.E. (to ATR_0):

1. Lusin's Separation Theorem: Any two disjoint analytic sets can be separated by a Borel set. (Simpson)
2. Mahlo's Theorem: Given any two countable closed compact subsets of the reals, one can be homeomorphically embedded in the other. (Friedman and Hirst)
3. Sherman's Inequality: If α, β, and γ are countable well orderings, then

$$(\alpha + \beta)\gamma \leq \alpha\gamma + \beta\gamma \qquad \text{(Hirst)}$$

Π^1_1 comprehension

The system $\Pi^1_1 - \mathsf{CA}_0$ is RCA_0 plus the axioms asserting the existence of the set $\{n \in \mathbb{N} \mid \theta(n)\}$ for $\theta \in \Pi^1_1$. (That is, θ has one universal set quantifier and no other set quantifiers.)

A tool and graph theory

Thm (RCA_0) T.F.A.E. (to $\Pi^1_1 - \mathsf{CA}_0$):

1. If $\langle T_i \rangle_{n\in\mathbb{N}}$ is a sequence of trees then there is a function $f : \mathbb{N} \to 2$ such that $f(n) = 1$ iff T_n is well founded.

2. For any graph H, and any sequence of graphs $\langle G_i \rangle_{i\in\mathbb{N}}$, there is a function $f : \mathbb{N} \to 2$ such that $f(n) = 1$ iff H is isomorphic to a subgraph of G. (Hirst and Lempp)

A few references

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