Nonuniformity in Reverse Mathematics

Jeffry L. Hirst

Department of Mathematical Sciences
Appalachian State University

(in collaboration with Carl Mummert, U. of Michigan)

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www.mathsci.appstate.edu/~jlh

Reverse Mathematics

Goal: Determine what set existence axioms are needed to prove familiar theorems.

Method: Prove results of the form

$$\mathsf{RCA}_0 \vdash \mathbf{AX} \leftrightarrow \mathbf{THM}$$

where:

- RCA_0 is a weak axiom system,
- **AX** is a set existence axiom selected from a small hierarchy of axioms, and
- **THM** is a familiar theorem.

$\mathsf{RCA_0}$: Recursive Comprehension

Language:

Integer variables: x, y, z Set variables: X, Y, Z

Axioms:

basic arithmetic axioms
(0, 1, +, ×, =, and < behave as usual.)

Restricted induction
$(\psi(0) \wedge \forall n(\psi(n) \rightarrow \psi(n+1))) \rightarrow \forall n \psi(n)$
where $\psi(n)$ has (at most) one x quantifier.

Recursive set comprehension
If $\theta \in \Sigma^0_1$ and $\psi \in \Pi^0_1$, and $\forall n(\theta(n) \leftrightarrow \psi(n))$, then there is a set X such that $\forall n(n \in X \leftrightarrow \theta(n))$

What can RCA_0 prove?

Arithmetic needed for coding.

Lots of finite graph theory, e.g.

Theorem 1. (RCA_0) *Every finite graph with no odd cycles is bipartite.*

A little analysis, e.g.

Theorem 2. (RCA_0) *If $\langle I_n \rangle_{n \in \mathbb{N}}$ is a sequence of nested real intervals, then there is a real number in their intersection.*

What can RCA_0 prove?

A little infinite graph theory:

Theorem 3. (RCA_0) *Every 2-regular graph with no odd cycles and exactly two connected components can be 2-colored.*

Proof. Given one designated vertex in each connected component, the 2-coloring can be "computed." □

Weak König's Lemma

Statement: Big very skinny trees are tall.

More formally: If T is an infinite tree in which each node is labeled 0 or 1, then T contains an infinite path.

WKL_0 is RCA_0 plus Weak König's Lemma.

Note: $\mathsf{RCA}_0 \nvdash \mathsf{WKL}_0$

Some reverse mathematics!

Theorem 4. (RCA_0) *The following are equivalent:*

1. WKL_0
2. *If* $\langle G_i \rangle_{i \in \mathbb{N}}$ *is an infinite sequence of infinite* 2*-regular graphs each of which has no odd cycles and exactly two connected components, then there is a sequence* $\langle f_i \rangle_{i \in \mathbb{N}}$ *of functions such that for each* i*,* f_i *is a* 2*-coloring of* G_i*.*

The proof of $1 \rightarrow 2$ consists of a construction of an infinite $0-1$ tree such that any path through the tree codes all the desired 2-colorings. The converse is (perhaps) more entertaining...

Toward the reversal:

Theorem 5. (RCA_0) *The following are equivalent:*

1. WKL_0
2. *If f and g are one to one functions with disjoint ranges, then there is a set X such that for all x, $f(x) \in X$ and $g(x) \notin X$.*

To prove that $2 \rightarrow 1$, we work in RCA_0 and use the statement about sequences of graphs to deduce the existence of a separating set.

Suppose that f and g are injections with disjoint ranges, $f(2) = 0$, $g(1) = 1$, and 2 is in the range of neither function. Build G_0, G_1, and G_2 as follows:

Nonuniformity

RCA_0 proves our statement about 2-colorings for a single graph, but does not prove the statement for infinite sequences of graphs.

In general, we are interested in situations where

$$\mathsf{RCA}_0 \vdash \forall X \exists Y \theta(X, Y)$$

but

$$\mathsf{RCA}_0 \nvdash \forall \langle X_i \rangle \ \exists \langle Y_i \rangle \ \forall n \theta(X_n, Y_n).$$

Encoding the reals

A *real number* is a function $x : \mathbb{N} \longrightarrow \mathbb{Q}$ such that

$$\forall k \forall i \; |x(k) - x(k+i)| \leq 2^{-k}$$

(that is, $\langle x(i) \rangle_{i \in \mathbb{N}}$ is a rapidly converging Cauchy sequence of rationals.)

Examples of reals

$\sqrt{2}$: $\quad 1, 1.4, 1.41, 1.414, 1.4142, \ldots$

π : $\quad 3, 3.1, 3.14, 3.141, 3.1415, \ldots$

0 : $\quad 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

0 : $\quad 0, 0, 0, 0, 0, \ldots$

Relationships between reals

$x = y$ means: $\forall k \; |x(k) - y(k)| \leq 2^{-k+1}$

$x \leq y$ means: $\forall k \; (x(k) \leq y(k) + 2^{-k+1})$

$y < x$ means $x \not\leq y$,
which is $\exists k \; (y(k) + 2^{-k+1} < x(k))$

Theorem 6. (RCA_0) *If $\langle x_i \rangle_{i \leq n}$ is a finite sequence of reals, then there is a $j \leq n$ such that x_j is the minimum of the sequence.*

Theorem 7. (RCA_0) *The following are equivalent:*

1. WKL_0
2. *If $\langle x_k \rangle_{k \in \mathbb{N}}$ is a sequence of reals, then there is a sequence of integers $\langle \mu_k \rangle_{k \in \mathbb{N}}$ such that*

$$\forall k (x_{\mu_k} = \min\{x_j \mid j \leq k\}).$$

Sketch of (2) implies $\mathsf{WKL_0}$

Suppose f and g are injections with disjoint ranges. Use a sequence of indices of minima to construct a separating set.

If $f(3) = 0$, g(2)=1, and $2 \notin \mathsf{Ran} f \cup \mathsf{Ran} g$, build:

$$
\begin{array}{lrrrrl}
x_{0,f}: & 0 & 0 & 0 & -.0001 & \ldots \\
x_{0,g}: & 0 & 0 & 0 & 0 & \ldots \\
x_{1,f}: & -1 & -1 & -1 & -1 & \ldots \\
x_{1,g}: & -1 & -1 & -1.001 & -1.001 & \ldots \\
x_{2,f}: & -2 & -2 & -2 & -2 & \ldots \\
x_{2,g}: & -2 & -2 & -2 & -2 & \ldots
\end{array}
$$

A stronger axiom system: Arithmetical Comprehension

$\mathsf{ACA_0}$ is $\mathsf{RCA_0}$ plus the following comprehension scheme:

> For any formula $\theta(n)$ with only number quantifiers, the set $\{n \in \mathbb{N} \mid \theta(n)\}$ exists.

Note: $\mathsf{WKL_0} \nvdash \mathsf{ACA_0}$, but $\mathsf{ACA_0} \vdash \mathsf{WKL_0}$

The tool:

Theorem 8. ($\mathsf{RCA_0}$) *The following are equivalent:*

1. $\mathsf{ACA_0}$

2. If $f : \mathbb{N} \to \mathbb{N}$ *is 1-1, then* $Ran(f)$ *exists.*

Two forms of Dedekind cuts

Lower Dedekind cuts: a set $\emptyset \subsetneq \lambda \subsetneq \mathbb{Q}$ such that

$$\forall s \in \mathbb{Q} \forall s' \in \mathbb{Q} \left((s \in \lambda \land s' \notin \lambda) \to s < s' \right).$$

Open Dedekind cuts: a lower Dedekind cut σ with no greatest element.

Theorem 9. (RCA_0) *Every Dedekind cut is equal to an open cut.*

Theorem 10. (RCA_0) *The following are equivalent:*

1. ACA_0.
2. *If* $\langle \lambda_i \rangle_{i \in \mathbb{N}}$ *is a sequence of Dedekind cuts, then there is a sequence* $\langle \sigma_i \rangle_{i \in \mathbb{N}}$ *of open cuts such that for every* $i \in \mathbb{N}$, $\lambda_i = \sigma_i$.

Part of the proof of Theorem 10

We want: *Dedekind cuts* $\to$ *open cuts* implies ACA_0.

Suppose $f : \mathbb{N}^+ \to \mathbb{N}$ is an injection. We'll find its range.

Define the sequence $\langle \lambda_i \rangle_{i \in \mathbb{N}}$ of Dedekind cuts by putting $q \in \mathbb{Q}$ in λ_i if and only if:

$$q \leq 0 \quad \text{or} \quad q > 0 \text{ and } (\exists t < 1/q)(f(t) = i).$$

Informally,

if $i \notin \mathrm{Range}(f)$, then $\lambda_i = (-\infty, 0] \cap \mathbb{Q}$, and

if $f(t) = i$, then $\lambda_i = (-\infty, 1/t) \cap \mathbb{Q}$.

If σ_i is an open cut with $\sigma_i = \lambda_i$, then $i \in \mathrm{Range}(f)$ if and only if $0 \in \sigma_i$.

Carl Mummert's nice example

Theorem 11. (RCA_0) *Every 2×2 real valued matrix has a Jordan decomposition.*

Theorem 12. (RCA_0) *The following are equivalent:*

1. ACA_0.
2. *Given a sequence of 2×2 real valued matrices, we can find the sequence of their Jordan forms.*

Carl and Jeff's conjecture

Conjecture 13. *If* $\widehat{\mathsf{HA}}^{\#}$ *proves a* Π^1_2 *statement of the form* $\forall A\, \exists B\, \Theta(A, B)$, *where* Θ *is arithmetical, then the uniformized statement*

$$\forall \langle A_n \rangle_{n \in \mathbb{N}}\ \exists \langle B_n \rangle_{n \in \mathbb{N}}\ \forall n\, \Theta(A_n, B_n)$$

is provable in RCA_0.

The contrapositive essentially asserts that if we can reverse the uniformized version of a Π^1_2 statement, then the original statement is not provable in an axiomatization of a substantial fragment of intuitionistic analysis.

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