Discrete Mathematics:

Venn Diagrams and Logic

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Outline

Venn Diagrams:

- Representing unions and intersections
- Venn diagrams and Eulerian diagrams
- Locating elements: Web-based materials
- Counting and cardinality
- Venn diagrams and reasoning
- Venn diagrams as graphs

Proof by induction

- The induction scheme and its variations
- Proving facts about natural numbers

Venn diagrams are used to represent sets and set operations.

Example:

Use Venn diagrams to illustrate sets A and B and…

1. $A \cup B$

2. $A \cap B$

3. $\bar{A} \cap B$

More than one shaded graph can be used to represent a particular situation.

Eulerian diagrams never have empty compartments.

Venn diagrams (as used by Venn) have compartments for every possibility. Empty compartments are shaded.

Most people use the term "Venn diagram" to refer to both Eulerian diagrams and Venn diagrams.

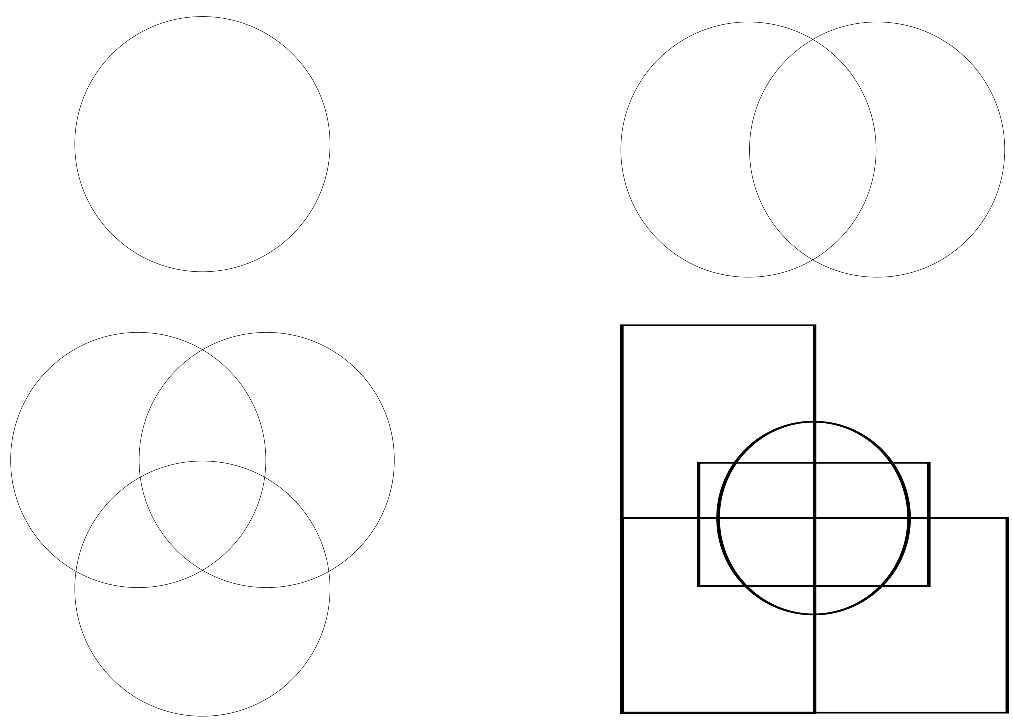
Example: Use an Eulerian diagram and a Venn diagram to represent the following sets: ducks, birds, mammals.

Here is a problem from Venn's *Symbolic Logic*:

> Every student must take Greek or Latin.
>
> Anyone who takes only one of Latin and Greek must take both English and French.
>
> Anyone who takes both Greek and Latin must take either English or French.

How many languages may a student take?

Venn Diagrams of various sizes



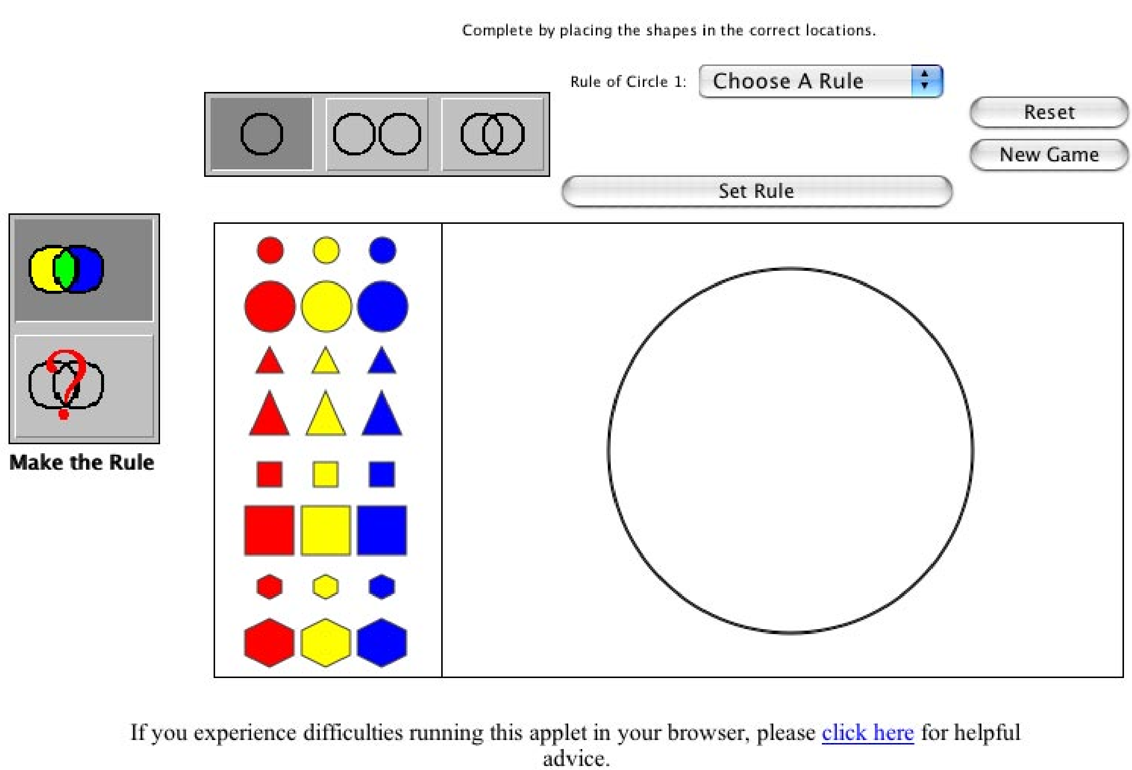
Elements in sets

Students can gain understanding of venn diagrams and sets by placing elements in the appropriate location in diagrams. Two Shodor Interactivate activities that based on this are:

Venn Diagrams:
http://www.shodor.org/master/interactivate/activities/vdiagram/index.html

Venn Diagram Shape Sorter:
http://www.shodor.org/master/interactivate/activities/venndia/index.html

You can also access these tools by following the path:
www.shodor.org → Master Tools → Project Interactivate → Teacher Resources → Activities Index



Here is a problem from an NCTM sponsored site:
http://www.figurethis.org/challenges/c20/challenge.htm

Figure This! (Math Challenges for Families)
There are four basketball games tonight. Three sports writers predict the winners in the morning paper.

- Perimeter picks:
  Raptors, Pacers, Magic, and 76ers.
- Exponent picks:
  Hawks, Pistons, Magic, and Raptors.
- Helix picks:
  Heat, Pacers, Pistons, and Raptors
- No one picks the Bucks.

WHO PLAYED WHOM?

Counting problems

Venn diagrams can be used to solve problems involving cardinality of finite sets.

Example: Suppose there are 50 beads in a drawer: 25 are glass, 30 are red, 20 are spherical, 18 are red glass, 12 are glass spheres, 15 are red spheres, and 8 are red glass spheres. How many beads are neither red, nor glass, nor spheres?

It's possible to do these counting problems with systems of linear equations. Since we use a variable for each compartment in the Venn diagram, the systems tend to be pretty big.

Example: The 34 faculty members at a local college invested their retirement contributions in a stock fund and a money market fund. There are 22 faculty with money in the stock fund and 27 with money in the money market fund. How many faculty have money in both funds?

Here is a counting problem from *Finite Mathematics* by Weiss and Yoseloff:
500 people receive free samples of shampoo. Each person was called three times and asked if they were using the product. Let F be the set of people that said they were using the shampoo on the first call and define S and T similarly for the second and third calls. The following data was collected:

$\lvert F\rvert = 135$	$\lvert S\rvert = 198$	$\lvert T\rvert = 280$
$\lvert F \cap S\rvert = 70$	$\lvert F \cap T\rvert = 56$	$\lvert S \cap T\rvert = 123$
$\lvert F \cap S \cap T\rvert = 51$		

1. How many people received samples but never used them?

2. How many people started using the product between the second and third call?

3. How many people were using the product at the first call, but were not using it at the second and third calls?

Venn Diagrams and Reasoning

The concepts of implication and subset are closely related. "If $x \in P$ then $x \in Q$" means the same thing as $P \subseteq Q$.

If we use P as shorthand for $x \in P$ and use $\rightarrow$ to represent implication, then

$$P \rightarrow Q \text{ translates as } P \subseteq Q.$$

Here are some standard rules of inference and their set theoretic translations.

Modus Ponens (Deductive reasoning)

From $P \rightarrow Q$ and P, deduce Q.

From $P \subseteq Q$ and $x \in P$, deduce $x \in Q$.

Modus Tollens (Inductive reasoning)

From $P \rightarrow Q$ and $\neg Q$, deduce $\neg P$.

From $P \subseteq Q$ and $x \in \bar{Q}$, then $x \in \bar{P}$.

Hypothetical Syllogism (Transitive reasoning)

From $P \rightarrow Q$ and $Q \rightarrow R$, deduce $P \rightarrow R$.

From $P \subseteq Q$ and $Q \subseteq R$, deduce $P \subseteq R$.

Here's a set of premises from Lewis Carroll's *Symbolic Logic*.

Babies are illogical.

No one is despised who can manage a crocodile.

Illogical persons are despised.

We can represent the premises with a (shaded) Venn diagram, and draw several conclusions.

More examples from Lewis Carroll:

1. All puddings are nice.
   This dish is a pudding.
   No nice things are wholesome.

2. There are no pencils of mine in this box.
   No sugar-plums of mine are cigars.
   The whole of my property, that is not in this box, consists of cigars.

Venn diagrams as graphs

For graph theorists, a Venn diagram for n sets $(A_1, A_2, \dots A_n)$ must contain exactly one region for each intersection of the form $B_1 \cap B_2 \cap \dots B_n$ where each B_i is either A_i or $\bar{A}_i$. (That is, graph theorists think like Venn.)

Theorem (Chilakamarri, Hamburger, and Pippert 1996) Every Venn diagram for n sets can be extended to a Venn diagram for $n+1$ sets.

A Venn diagram is *simple* if no three boundary curves intersect in a single point.

Winkler's Conjecture (1984) Every simple Venn diagram for n sets can be extended to a simple Venn diagram for $n+1$ sets.

References for Venn diagrams

John Venn, *Symbolic logic,* Lenox Hill, New York, 1971 (Original printing: 1894).

Lewis Carroll, *Symbolic logic and the game of logic,* Dover, New York, 1958 (Original printing: 1896).

N.A. Weiss and M.L. Yoseloff, *Finite mathematics,* Worth Publishers, New York, 1975.

Frank Ruskey, *A survey of venn diagrams,*
Electronic Journal of Combinatorics
http://www.combinatorics.org/Surveys/ds5/VennEJC.html

Shodor interactivate activities:
http://www.shodor.org/master/interactivate/

NCTM Figure this:
http://www.figurethis.org/challenges/c20/challenge.htm

NCTM Illuminations on Venn diagrams:
http://illuminations.nctm.org/lessonplans/9-12/reasoning/index.html

Proof by Induction

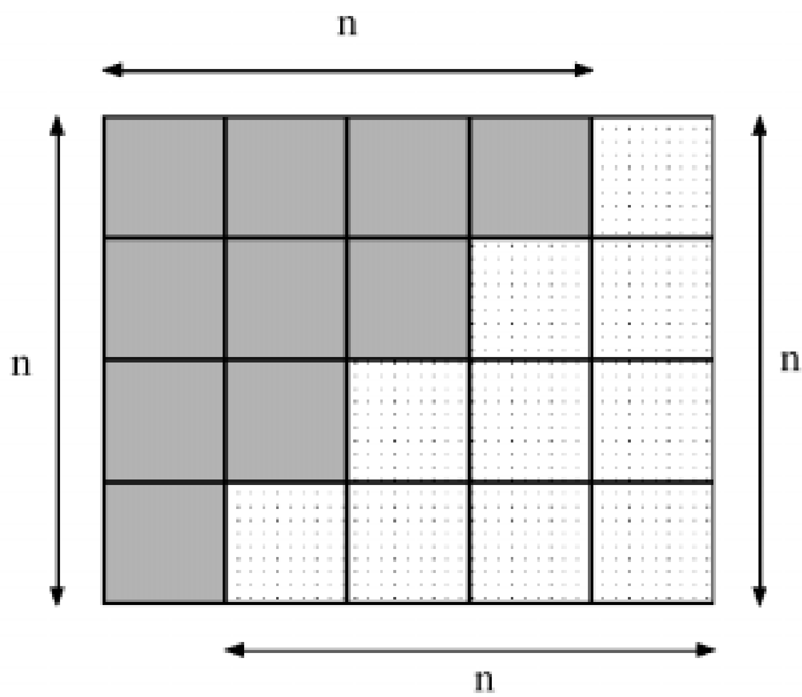
Suppose that $P(n)$ is a statement about natural numbers of the form "such and such a property holds for n." We can prove that $P(n)$ holds for all n using the following agenda:

1. **Prove $P(0)$.** (This is called the *base case*.)
2. **Prove that $P(n)$ implies $P(n+1)$.**
   (Usually we do this by assuming $P(n)$ and deducing $P(n+1)$, but we could use other reasoning techniques. This process is called the *induction step*, and $P(n)$ is called the induction hypothesis.)
3. **Conclude by induction that $P(n)$ holds for all n.**

Example: Prove the following childhood discovery of Gauss by induction:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

An alternative proof of $\sum_{j=1}^{n} j = n(n+1)/2$



Other good induction problems:

1. $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$

2. $\sum_{k=1}^{n} 2k = n^2 + n$

3. $k \geq 1 \rightarrow 8 | (9^k - 1)$ (Note: You can use 1 as the base case, and assume that n is always at least 1.)

4. $n \geq 3 \rightarrow n^2 \leq 5n!$ (Note: If $a \leq b$ and $c \leq d$, then $ac \leq bd$.)

5. The sum of the first n odd numbers is n^2. (Note: You may want to reformulate this using sequence notation.)

Why does induction work?

Two reasonable answers:

1. Induction is an axiom of formal Peano arithmetic. The natural numbers satisfy all the axioms of Peano arithmetic.

2. The natural numbers are well-ordered. This means that every nonempty subset of the natural numbers has a least element. Suppose that $P(n)$ fails to hold for some n. Then the set of natural numbers n such that $P(n)$ fails is nonempty and has a least element. Call that element a. Either $a = 0$, or $P(a-1)$ holds and $P(a)$ fails. Thus, either the base case is false or the induction step is false.

Strong induction

Here is another induction scheme:

1. Prove $P(0)$.

2. Prove that if $P(m)$ holds for all $m < n$, then $P(n)$ holds. (We get to use *all* the previous cases!)

3. Conclude by induction that $P(n)$ holds for all n.

Strong induction is great for proving facts about recurrence relations.

The Fibonacci sequence is a sequence of integers defined by the formulas $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.

Strong induction helps in proving the following:

1. Prove that for all $n \geq 1$, $f_n \leq 2^n$.
2. Prove that for all $n \geq 5$, $f_n \geq n$.
3. Prove that for all $n \geq 9$, $f_n \geq 3n$.

All these induction exercises (and many more) appear in Chapter 3 of *A primer for logic and proof* by Hirst and Hirst. A draft is available at:

http://www.mathsci.appstate.edu/~jlh/pdf/hirst.pdf

For more information about the Fibonacci sequence, see the Fibonacci Quarterly. (A journal that is available in the ASU library.)

These slides are available at:
http://www.mathsci.appstate.edu/~jlh/snp/pdfslides/venn.pdf